	SAT	ACT
Mean	1500	21
SD	300	5

Table 3.4: Mean and standard deviation for the SAT and ACT.

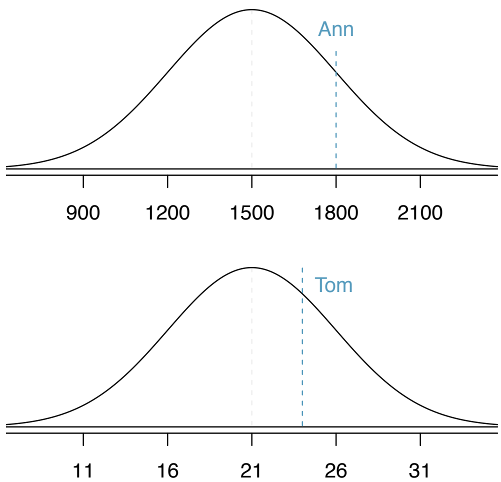

Figure 3.5: Ann's and Tom's scores shown with the distributions of SAT and ACT scores.

3.1.2 Standardizing with Z-scores

Example 3.2 Table 3.4 shows the mean and standard deviation for total scores on the SAT and ACT. The distribution of SAT and ACT scores are both nearly normal. Suppose Ann scored 1800 on her SAT and Tom scored 24 on his ACT. Who performed better?

We use the standard deviation as a guide. Ann is 1 standard deviation above average on the SAT: $1500 + 300 = 1800$. Tom is 0.6 standard deviations above the mean on the ACT: $21 + 0.6 \times 5 = 24$. In Figure 3.5, we can see that Ann tends to do better with respect to everyone else than Tom did, so her score was better.

Example 3.2 used a standardization technique called a Z-score, a method most commonly employed for nearly normal observations but that may be used with any distribution. The **Z-score** of an observation is defined as the number of standard deviations it falls above or below the mean. If the observation is one standard deviation above the mean, its Z-score is 1. If it is 1.5 standard deviations *below* the mean, then its Z-score is -1.5. If x is an observation from a distribution $N(\mu, \sigma)$, we define the Z-score mathematically as

Z
Z-score, the standardized observation

$$Z = \frac{x - \mu}{\sigma}$$

Using $\mu_{SAT} = 1500$, $\sigma_{SAT} = 300$, and $x_{Ann} = 1800$, we find Ann's Z-score:

$$Z_{Ann} = \frac{x_{Ann} - \mu_{SAT}}{\sigma_{SAT}} = \frac{1800 - 1500}{300} = 1$$

The Z-score

The Z-score of an observation is the number of standard deviations it falls above or below the mean. We compute the Z-score for an observation x that follows a distribution with mean μ and standard deviation σ using

$$Z = \frac{x - \mu}{\sigma}$$

Guided Practice 3.3 Use Tom's ACT score, 24, along with the ACT mean and standard deviation to compute his Z-score.[3]

Observations above the mean always have positive Z-scores while those below the mean have negative Z-scores. If an observation is equal to the mean (e.g. SAT score of 1500), then the Z-score is 0.

Guided Practice 3.4 Let X represent a random variable from $N(\mu = 3, \sigma = 2)$, and suppose we observe $x = 5.19$. (a) Find the Z-score of x. (b) Use the Z-score to determine how many standard deviations above or below the mean x falls.[4]

Guided Practice 3.5 Head lengths of brushtail possums follow a nearly normal distribution with mean 92.6 mm and standard deviation 3.6 mm. Compute the Z-scores for possums with head lengths of 95.4 mm and 85.8 mm.[5]

We can use Z-scores to roughly identify which observations are more unusual than others. One observation x_1 is said to be more unusual than another observation x_2 if the absolute value of its Z-score is larger than the absolute value of the other observation's Z-score: $|Z_1| > |Z_2|$. This technique is especially insightful when a distribution is symmetric.

Guided Practice 3.6 Which of the observations in Guided Practice 3.5 is more unusual?[6]

3.1.3 Normal probability table

Example 3.7 Ann from Example 3.2 earned a score of 1800 on her SAT with a corresponding $Z = 1$. She would like to know what percentile she falls in among all SAT test-takers.

Ann's **percentile** is the percentage of people who earned a lower SAT score than Ann. We shade the area representing those individuals in Figure 3.6. The total area under the normal curve is always equal to 1, and the proportion of people who scored below Ann on the SAT is equal to the *area* shaded in Figure 3.6: 0.8413. In other words, Ann is in the 84^{th} percentile of SAT takers.

[3] $Z_{Tom} = \frac{x_{Tom} - \mu_{ACT}}{\sigma_{ACT}} = \frac{24-21}{5} = 0.6$

[4] (a) Its Z-score is given by $Z = \frac{x-\mu}{\sigma} = \frac{5.19-3}{2} = 2.19/2 = 1.095$. (b) The observation x is 1.095 standard deviations *above* the mean. We know it must be above the mean since Z is positive.

[5] For $x_1 = 95.4$ mm: $Z_1 = \frac{x_1-\mu}{\sigma} = \frac{95.4-92.6}{3.6} = 0.78$. For $x_2 = 85.8$ mm: $Z_2 = \frac{85.8-92.6}{3.6} = -1.89$.

[6] Because the *absolute value* of Z-score for the second observation is larger than that of the first, the second observation has a more unusual head length.

3 | PROBABILITY TOPICS

Figure 3.1 Meteor showers are rare, but the probability of them occurring can be calculated. (credit: Navicore/flickr)

Introduction

Chapter Objectives

By the end of this chapter, the student should be able to:

- Understand and use the terminology of probability.
- Determine whether two events are mutually exclusive and whether two events are independent.
- Calculate probabilities using the Addition Rules and Multiplication Rules.
- Construct and interpret Contingency Tables.
- Construct and interpret Venn Diagrams.
- Construct and interpret Tree Diagrams.

It is often necessary to "guess" about the outcome of an event in order to make a decision. Politicians study polls to guess their likelihood of winning an election. Teachers choose a particular course of study based on what they think students can comprehend. Doctors choose the treatments needed for various diseases based on their assessment of likely results. You may have visited a casino where people play games chosen because of the belief that the likelihood of winning is good. You may have chosen your course of study based on the probable availability of jobs.

You have, more than likely, used probability. In fact, you probably have an intuitive sense of probability. Probability deals with the chance of an event occurring. Whenever you weigh the odds of whether or not to do your homework or to study for an exam, you are using probability. In this chapter, you will learn how to solve probability problems using a systematic approach.

Collaborative Exercise

Your instructor will survey your class. Count the number of students in the class today.

- Raise your hand if you have any change in your pocket or purse. Record the number of raised hands.

- Raise your hand if you rode a bus within the past month. Record the number of raised hands.
- Raise your hand if you answered "yes" to BOTH of the first two questions. Record the number of raised hands.

Use the class data as estimates of the following probabilities. *P*(change) means the probability that a randomly chosen person in your class has change in his/her pocket or purse. *P*(bus) means the probability that a randomly chosen person in your class rode a bus within the last month and so on. Discuss your answers.

- Find *P*(change).
- Find *P*(bus).
- Find *P*(change AND bus). Find the probability that a randomly chosen student in your class has change in his/her pocket or purse and rode a bus within the last month.
- Find *P*(change|bus). Find the probability that a randomly chosen student has change given that he or she rode a bus within the last month. Count all the students that rode a bus. From the group of students who rode a bus, count those who have change. The probability is equal to those who have change and rode a bus divided by those who rode a bus.

3.1 | Terminology

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity. An **experiment** is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a **chance** experiment. Flipping one fair coin twice is an example of an experiment.

A result of an experiment is called an **outcome**. The **sample space** of an experiment is the set of all possible outcomes. Three ways to represent a sample space are: to list the possible outcomes, to create a tree diagram, or to create a Venn diagram. The uppercase letter *S* is used to denote the sample space. For example, if you flip one fair coin, $S = \{H, T\}$ where H = heads and T = tails are the outcomes.

An **event** is any combination of outcomes. Upper case letters like *A* and *B* represent events. For example, if the experiment is to flip one fair coin, event *A* might be getting at most one head. The probability of an event *A* is written $P(A)$.

The **probability** of any outcome is the **long-term relative frequency** of that outcome. **Probabilities are between zero and one, inclusive** (that is, zero and one and all numbers between these values). $P(A) = 0$ means the event *A* can never happen. $P(A) = 1$ means the event *A* always happens. $P(A) = 0.5$ means the event *A* is equally likely to occur or not to occur. For example, if you flip one fair coin repeatedly (from 20 to 2,000 to 20,000 times) the relative frequency of heads approaches 0.5 (the probability of heads).

Equally likely means that each outcome of an experiment occurs with equal probability. For example, if you toss a **fair**, six-sided die, each face (1, 2, 3, 4, 5, or 6) is as likely to occur as any other face. If you toss a fair coin, a Head (*H*) and a Tail (*T*) are equally likely to occur. If you randomly guess the answer to a true/false question on an exam, you are equally likely to select a correct answer or an incorrect answer.

To calculate the probability of an event *A* when all outcomes in the sample space are equally likely, count the number of outcomes for event *A* and divide by the total number of outcomes in the sample space. For example, if you toss a fair dime and a fair nickel, the sample space is $\{HH, TH, HT, TT\}$ where T = tails and H = heads. The sample space has four outcomes. A = getting one head. There are two outcomes that meet this condition $\{HT, TH\}$, so $P(A) = \frac{2}{4} = 0.5$.

Suppose you roll one fair six-sided die, with the numbers {1, 2, 3, 4, 5, 6} on its faces. Let event E = rolling a number that is at least five. There are two outcomes {5, 6}. $P(E) = \frac{2}{6}$. If you were to roll the die only a few times, you would not be surprised if your observed results did not match the probability. If you were to roll the die a very large number of times, you would expect that, overall, $\frac{2}{6}$ of the rolls would result in an outcome of "at least five". You would not expect exactly $\frac{2}{6}$. The long-term relative frequency of obtaining this result would approach the theoretical probability of $\frac{2}{6}$ as the number of repetitions grows larger and larger.

This important characteristic of probability experiments is known as the **law of large numbers** which states that as the number of repetitions of an experiment is increased, the relative frequency obtained in the experiment tends to become closer and closer to the theoretical probability. Even though the outcomes do not happen according to any set pattern or order, overall, the long-term observed relative frequency will approach the theoretical probability. (The word **empirical** is often used instead of the word observed.)

It is important to realize that in many situations, the outcomes are not equally likely. A coin or die may be **unfair**, or **biased**. Two math professors in Europe had their statistics students test the Belgian one Euro coin and discovered that in 250 trials, a head was obtained 56% of the time and a tail was obtained 44% of the time. The data seem to show that the coin is not a fair coin; more repetitions would be helpful to draw a more accurate conclusion about such bias. Some dice may be biased. Look at the dice in a game you have at home; the spots on each face are usually small holes carved out and then painted to make the spots visible. Your dice may or may not be biased; it is possible that the outcomes may be affected by the slight weight differences due to the different numbers of holes in the faces. Gambling casinos make a lot of money depending on outcomes from rolling dice, so casino dice are made differently to eliminate bias. Casino dice have flat faces; the holes are completely filled with paint having the same density as the material that the dice are made out of so that each face is equally likely to occur. Later we will learn techniques to use to work with probabilities for events that are not equally likely.

"OR" Event:

An outcome is in the event A OR B if the outcome is in A or is in B or is in both A and B. For example, let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$. A OR $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Notice that 4 and 5 are NOT listed twice.

"AND" Event:

An outcome is in the event A AND B if the outcome is in both A and B at the same time. For example, let A and B be $\{1, 2, 3, 4, 5\}$ and $\{4, 5, 6, 7, 8\}$, respectively. Then A AND $B = \{4, 5\}$.

The **complement** of event A is denoted A' (read "A prime"). A' consists of all outcomes that are **NOT** in A. Notice that $P(A) + P(A') = 1$. For example, let $S = \{1, 2, 3, 4, 5, 6\}$ and let $A = \{1, 2, 3, 4\}$. Then, $A' = \{5, 6\}$. $P(A) = \frac{4}{6}$, $P(A') = \frac{2}{6}$, and

$P(A) + P(A') = \frac{4}{6} + \frac{2}{6} = 1$

The **conditional probability** of A given B is written $P(A|B)$. $P(A|B)$ is the probability that event A will occur given that the event B has already occurred. **A conditional reduces the sample space**. We calculate the probability of A from the reduced sample space B. The formula to calculate $P(A|B)$ is $P(A|B) = \frac{P(A \text{AND} B)}{P(B)}$ where $P(B)$ is greater than zero.

For example, suppose we toss one fair, six-sided die. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let A = face is 2 or 3 and B = face is even (2, 4, 6). To calculate $P(A|B)$, we count the number of outcomes 2 or 3 in the sample space $B = \{2, 4, 6\}$. Then we divide that by the number of outcomes B (rather than S).

We get the same result by using the formula. Remember that S has six outcomes.

$$P(A|B) = \frac{P(A \text{AND} B)}{P(B)} = \frac{\frac{(\text{the number of outcomes that are 2 or 3 and even in} S)}{6}}{\frac{(\text{the number of outcomes that are even in} S)}{6}} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

Understanding Terminology and Symbols

It is important to read each problem carefully to think about and understand what the events are. Understanding the wording is the first very important step in solving probability problems. Reread the problem several times if necessary. Clearly identify the event of interest. Determine whether there is a condition stated in the wording that would indicate that the probability is conditional; carefully identify the condition, if any.

Example 3.1

The sample space S is the whole numbers starting at one and less than 20.

a. S = ______________________________
Let event A = the even numbers and event B = numbers greater than 13.

b. A = _____________________, B = _____________________
c. $P(A)$ = _____________, $P(B)$ = ________________
d. A AND B = ____________________, A OR B = ________________
e. $P(A \text{ AND } B)$ = _________, $P(A \text{ OR } B)$ = _____________
f. A' = _____________, $P(A')$ = _____________
g. $P(A) + P(A')$ = ____________
h. $P(A|B)$ = ___________, $P(B|A)$ = _____________; are the probabilities equal?

Solution 3.1

a. S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19}
b. A = {2, 4, 6, 8, 10, 12, 14, 16, 18}, B = {14, 15, 16, 17, 18, 19}
c. $P(A) = \frac{9}{19}$, $P(B) = \frac{6}{19}$
d. A AND B = {14,16,18}, A OR B = {2, 4, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19}
e. $P(A \text{ AND } B) = \frac{3}{19}$, $P(A \text{ OR } B) = \frac{12}{19}$
f. A' = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19; $P(A') = \frac{10}{19}$
g. $P(A) + P(A') = 1$ $\left(\frac{9}{19} + \frac{10}{19} = 1\right)$
h. $P(A|B) = \frac{P(A \text{AND} B)}{P(B)} = \frac{3}{6}$, $P(B|A) = \frac{P(A \text{AND} B)}{P(A)} = \frac{3}{9}$, No

Try It Σ

3.1 The sample space S is all the ordered pairs of two whole numbers, the first from one to three and the second from one to four (Example: (1, 4)).

a. S = ____________________________

Let event A = the sum is even and event B = the first number is prime.

b. A = _____________________, B = _____________________
c. $P(A)$ = _____________, $P(B)$ = ________________
d. A AND B = ____________________, A OR B = ________________
e. $P(A \text{ AND } B)$ = _________, $P(A \text{ OR } B)$ = _____________
f. B' = _____________, $P(B')$ = _____________
g. $P(A) + P(A')$ = ____________
h. $P(A|B)$ = ___________, $P(B|A)$ = _____________; are the probabilities equal?

Example 3.2

A fair, six-sided die is rolled. Describe the sample space S, identify each of the following events with a subset of S and compute its probability (an outcome is the number of dots that show up).

a. Event T = the outcome is two.
b. Event A = the outcome is an even number.
c. Event B = the outcome is less than four.
d. The complement of A.
e. A GIVEN B
f. B GIVEN A
g. A AND B
h. A OR B
i. A OR B'
j. Event N = the outcome is a prime number.
k. Event I = the outcome is seven.

Solution 3.2

a. $T = \{2\}$, $P(T) = \frac{1}{6}$
b. $A = \{2, 4, 6\}$, $P(A) = \frac{1}{2}$
c. $B = \{1, 2, 3\}$, $P(B) = \frac{1}{2}$
d. $A' = \{1, 3, 5\}$, $P(A') = \frac{1}{2}$
e. $A|B = \{2\}$, $P(A|B) = \frac{1}{3}$
f. $B|A = \{2\}$, $P(B|A) = \frac{1}{3}$
g. $A \text{ AND } B = \{2\}$, $P(A \text{ AND } B) = \frac{1}{6}$
h. $A \text{ OR } B = \{1, 2, 3, 4, 6\}$, $P(A \text{ OR } B) = \frac{5}{6}$
i. $A \text{ OR } B' = \{2, 4, 5, 6\}$, $P(A \text{ OR } B') = \frac{2}{3}$
j. $N = \{2, 3, 5\}$, $P(N) = \frac{1}{2}$
k. A six-sided die does not have seven dots. $P(7) = 0$.

3.2 | Independent and Mutually Exclusive Events

Independent and mutually exclusive do **not** mean the same thing.

Independent Events

Two events are independent if the following are true:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \text{ AND } B) = P(A)P(B)$

Two events *A* and *B* are **independent** if the knowledge that one occurred does not affect the chance the other occurs. For example, the outcomes of two roles of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. To show two events are independent, you must show **only one** of the above conditions. If two events are NOT independent, then we say that they are **dependent**.

Sampling may be done **with replacement** or **without replacement**.

- **With replacement**: If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be independent, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement**: When sampling is done without replacement, each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be dependent or not independent.

If it is not known whether *A* and *B* are independent or dependent, **assume they are dependent until you can show otherwise**.

Example 3.4

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *J* (jack), *Q* (queen), *K* (king) of that suit.

a. Sampling with replacement:
Suppose you pick three cards with replacement. The first card you pick out of the 52 cards is the *Q* of spades. You put this card back, reshuffle the cards and pick a second card from the 52-card deck. It is the ten of clubs. You put this card back, reshuffle the cards and pick a third card from the 52-card deck. This time, the card is the *Q* of spades again. Your picks are {*Q* of spades, ten of clubs, *Q* of spades}. You have picked the *Q* of spades twice. You pick each card from the 52-card deck.

b. Sampling without replacement:
Suppose you pick three cards without replacement. The first card you pick out of the 52 cards is the *K* of hearts. You put this card aside and pick the second card from the 51 cards remaining in the deck. It is the three of diamonds. You put this card aside and pick the third card from the remaining 50 cards in the deck. The third card is the *J* of spades. Your picks are {*K* of hearts, three of diamonds, *J* of spades}. Because you have picked the cards without replacement, you cannot pick the same card twice.

Try It Σ

3.4 You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *J* (jack), *Q* (queen), *K* (king) of that suit. Three cards are picked at random.

a. Suppose you know that the picked cards are *Q* of spades, *K* of hearts and *Q* of spades. Can you decide if the sampling was with or without replacement?

b. Suppose you know that the picked cards are *Q* of spades, *K* of hearts, and *J* of spades. Can you decide if the sampling was with or without replacement?

Example 3.5

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *J* (jack), *Q* (queen), and *K* (king) of that suit. *S* = spades, *H* = Hearts, *D* = Diamonds, *C* = Clubs.

a. Suppose you pick four cards, but do not put any cards back into the deck. Your cards are *QS*, 1*D*, 1*C*, *QD*.
b. Suppose you pick four cards and put each card back before you pick the next card. Your cards are *KH*, 7*D*, 6*D*, *KH*.

Which of a. or b. did you sample with replacement and which did you sample without replacement?

Solution 3.5

a. Without replacement; b. With replacement

Try It Σ

3.5 You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *J* (jack), *Q* (queen), and *K* (king) of that suit. *S* = spades, *H* = Hearts, *D* = Diamonds, *C* = Clubs. Suppose that you sample four cards without replacement. Which of the following outcomes are possible? Answer the same question for sampling with replacement.

a. *QS*, 1*D*, 1*C*, *QD*
b. *KH*, 7*D*, 6*D*, *KH*
c. *QS*, 7*D*, 6*D*, *KS*

Mutually Exclusive Events

A and *B* are **mutually exclusive** events if they cannot occur at the same time. This means that *A* and *B* do not share any outcomes and $P(A \text{ AND } B) = 0$.

For example, suppose the sample space *S* = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Let *A* = {1, 2, 3, 4, 5}, *B* = {4, 5, 6, 7, 8}, and *C* = {7, 9}. *A* AND *B* = {4, 5}. $P(A \text{ AND } B) = \frac{2}{10}$ and is not equal to zero. Therefore, *A* and *B* are not mutually exclusive. *A* and *C* do not have any numbers in common so $P(A \text{ AND } C) = 0$. Therefore, *A* and *C* are mutually exclusive.

If it is not known whether *A* and *B* are mutually exclusive, **assume they are not until you can show otherwise**. The following examples illustrate these definitions and terms.

Example 3.6

Flip two fair coins. (This is an experiment.)

The sample space is {*HH*, *HT*, *TH*, *TT*} where *T* = tails and *H* = heads. The outcomes are *HH*, *HT*, *TH*, and *TT*. The outcomes HT and TH are different. The *HT* means that the first coin showed heads and the second coin showed tails. The *TH* means that the first coin showed tails and the second coin showed heads.

- Let *A* = the event of getting **at most one tail**. (At most one tail means zero or one tail.) Then *A* can be written as {*HH*, *HT*, *TH*}. The outcome *HH* shows zero tails. *HT* and *TH* each show one tail.

- Let B = the event of getting all tails. B can be written as $\{TT\}$. B is the **complement** of A, so $B = A'$. Also, $P(A) + P(B) = P(A) + P(A') = 1$.
- The probabilities for A and for B are $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{4}$.
- Let C = the event of getting all heads. $C = \{HH\}$. Since $B = \{TT\}$, $P(B \text{ AND } C) = 0$. B and C are mutually exclusive. (B and C have no members in common because you cannot have all tails and all heads at the same time.)
- Let D = event of getting **more than one** tail. $D = \{TT\}$. $P(D) = \frac{1}{4}$
- Let E = event of getting a head on the first roll. (This implies you can get either a head or tail on the second roll.) $E = \{HT, HH\}$. $P(E) = \frac{2}{4}$
- Find the probability of getting **at least one** (one or two) tail in two flips. Let F = event of getting at least one tail in two flips. $F = \{HT, TH, TT\}$. $P(F) = \frac{3}{4}$

Try It Σ

3.6 Draw two cards from a standard 52-card deck with replacement. Find the probability of getting at least one black card.

Example 3.7

Flip two fair coins. Find the probabilities of the events.

a. Let F = the event of getting at most one tail (zero or one tail).
b. Let G = the event of getting two faces that are the same.
c. Let H = the event of getting a head on the first flip followed by a head or tail on the second flip.
d. Are F and G mutually exclusive?
e. Let J = the event of getting all tails. Are J and H mutually exclusive?

Solution 3.7

Look at the sample space in **Example 3.6**.

a. Zero (0) or one (1) tails occur when the outcomes HH, TH, HT show up. $P(F) = \frac{3}{4}$
b. Two faces are the same if HH or TT show up. $P(G) = \frac{2}{4}$
c. A head on the first flip followed by a head or tail on the second flip occurs when HH or HT show up. $P(H) = \frac{2}{4}$
d. F and G share HH so $P(F \text{ AND } G)$ is not equal to zero (0). F and G are not mutually exclusive.
e. Getting all tails occurs when tails shows up on both coins (TT). H's outcomes are HH and HT.

J and H have nothing in common so $P(J \text{ AND } H) = 0$. J and H are mutually exclusive.

Try It Σ

3.7 A box has two balls, one white and one red. We select one ball, put it back in the box, and select a second ball (sampling with replacement). Find the probability of the following events:

a. Let *F* = the event of getting the white ball twice.

b. Let *G* = the event of getting two balls of different colors.

c. Let *H* = the event of getting white on the first pick.

d. Are *F* and *G* mutually exclusive?

e. Are *G* and *H* mutually exclusive?

Example 3.8

Roll one fair, six-sided die. The sample space is {1, 2, 3, 4, 5, 6}. Let event *A* = a face is odd. Then *A* = {1, 3, 5}. Let event *B* = a face is even. Then *B* = {2, 4, 6}.

- Find the complement of *A*, *A′*. The complement of *A*, *A′*, is *B* because *A* and *B* together make up the sample space. $P(A) + P(B) = P(A) + P(A') = 1$. Also, $P(A) = \frac{3}{6}$ and $P(B) = \frac{3}{6}$.
- Let event *C* = odd faces larger than two. Then *C* = {3, 5}. Let event *D* = all even faces smaller than five. Then *D* = {2, 4}. $P(C \text{ AND } D) = 0$ because you cannot have an odd and even face at the same time. Therefore, *C* and *D* are mutually exclusive events.
- Let event *E* = all faces less than five. *E* = {1, 2, 3, 4}.

Are *C* and *E* mutually exclusive events? (Answer yes or no.) Why or why not?

Solution 3.8

No. *C* = {3, 5} and *E* = {1, 2, 3, 4}. $P(C \text{ AND } E) = \frac{1}{6}$. To be mutually exclusive, $P(C \text{ AND } E)$ must be zero.

- Find $P(C|A)$. This is a conditional probability. Recall that the event *C* is {3, 5} and event *A* is {1, 3, 5}. To find $P(C|A)$, find the probability of *C* using the sample space *A*. You have reduced the sample space from the original sample space {1, 2, 3, 4, 5, 6} to {1, 3, 5}. So, $P(C|A) = \frac{2}{3}$.

Try It Σ

3.8 Let event *A* = learning Spanish. Let event *B* = learning German. Then *A* AND *B* = learning Spanish and German. Suppose $P(A) = 0.4$ and $P(B) = 0.2$. $P(A \text{ AND } B) = 0.08$. Are events *A* and *B* independent? Hint: You must show ONE of the following:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \text{ AND } B) = P(A)P(B)$

Example 3.9

Let event *G* = taking a math class. Let event *H* = taking a science class. Then, *G* AND *H* = taking a math class and a science class. Suppose $P(G) = 0.6$, $P(H) = 0.5$, and $P(G \text{ AND } H) = 0.3$. Are *G* and *H* independent?

If G and H are independent, then you must show **ONE** of the following:

- $P(G|H) = P(G)$
- $P(H|G) = P(H)$
- $P(G \text{ AND } H) = P(G)P(H)$

NOTE

The choice you make depends on the information you have. You could choose any of the methods here because you have the necessary information.

a. Show that $P(G|H) = P(G)$.

Solution 3.9

$$P(G|H) = \frac{P(G \text{ AND } H)}{P(H)} = \frac{0.3}{0.5} = 0.6 = P(G)$$

b. Show $P(G \text{ AND } H) = P(G)P(H)$.

Solution 3.9

$P(G)P(H) = (0.6)(0.5) = 0.3 = P(G \text{ AND } H)$

Since G and H are independent, knowing that a person is taking a science class does not change the chance that he or she is taking a math class. If the two events had not been independent (that is, they are dependent) then knowing that a person is taking a science class would change the chance he or she is taking math. For practice, show that $P(H|G) = P(H)$ to show that G and H are independent events.

Try It Σ

3.9 In a bag, there are six red marbles and four green marbles. The red marbles are marked with the numbers 1, 2, 3, 4, 5, and 6. The green marbles are marked with the numbers 1, 2, 3, and 4.

- R = a red marble
- G = a green marble
- O = an odd-numbered marble
- The sample space is $S = \{R1, R2, R3, R4, R5, R6, G1, G2, G3, G4\}$.

S has ten outcomes. What is $P(G \text{ AND } O)$?

Example 3.10

Let event C = taking an English class. Let event D = taking a speech class.

Suppose $P(C) = 0.75$, $P(D) = 0.3$, $P(C|D) = 0.75$ and $P(C \text{ AND } D) = 0.225$.

Justify your answers to the following questions numerically.

a. Are C and D independent?

b. Are C and D mutually exclusive?

c. What is $P(D|C)$?

Solution 3.10

a. Yes, because $P(C|D) = P(C)$.

b. No, because $P(C \text{ AND } D)$ is not equal to zero.

c. $P(D|C) = \dfrac{P(C \text{ AND } D)}{P(C)} = \dfrac{0.225}{0.75} = 0.3$

Try It Σ

3.10 A student goes to the library. Let events B = the student checks out a book and D = the student checks out a DVD. Suppose that $P(B) = 0.40$, $P(D) = 0.30$ and $P(B \text{ AND } D) = 0.20$.

a. Find $P(B|D)$.

b. Find $P(D|B)$.

c. Are B and D independent?

d. Are B and D mutually exclusive?

Example 3.11

In a box there are three red cards and five blue cards. The red cards are marked with the numbers 1, 2, and 3, and the blue cards are marked with the numbers 1, 2, 3, 4, and 5. The cards are well-shuffled. You reach into the box (you cannot see into it) and draw one card.

Let R = red card is drawn, B = blue card is drawn, E = even-numbered card is drawn.

The sample space S = $R1$, $R2$, $R3$, $B1$, $B2$, $B3$, $B4$, $B5$. S has eight outcomes.

- $P(R) = \frac{3}{8}$. $P(B) = \frac{5}{8}$. $P(R \text{ AND } B) = 0$. (You cannot draw one card that is both red and blue.)
- $P(E) = \frac{3}{8}$. (There are three even-numbered cards, $R2$, $B2$, and $B4$.)
- $P(E|B) = \frac{2}{5}$. (There are five blue cards: $B1$, $B2$, $B3$, $B4$, and $B5$. Out of the blue cards, there are two even cards; $B2$ and $B4$.)
- $P(B|E) = \frac{2}{3}$. (There are three even-numbered cards: $R2$, $B2$, and $B4$. Out of the even-numbered cards, to are blue; $B2$ and $B4$.)
- The events R and B are mutually exclusive because $P(R \text{ AND } B) = 0$.
- Let G = card with a number greater than 3. $G = \{B4, B5\}$. $P(G) = \frac{2}{8}$. Let H = blue card numbered between one and four, inclusive. $H = \{B1, B2, B3, B4\}$. $P(G|H) = \frac{1}{4}$. (The only card in H that has a number greater than three is $B4$.) Since $\frac{2}{8} = \frac{1}{4}$, $P(G) = P(G|H)$, which means that G and H are independent.

Try It Σ

3.11 In a basketball arena,

- 70% of the fans are rooting for the home team.
- 25% of the fans are wearing blue.
- 20% of the fans are wearing blue and are rooting for the away team.
- Of the fans rooting for the away team, 67% are wearing blue.

Let A be the event that a fan is rooting for the away team.
Let B be the event that a fan is wearing blue.
Are the events of rooting for the away team and wearing blue independent? Are they mutually exclusive?

Example 3.12

In a particular college class, 60% of the students are female. Fifty percent of all students in the class have long hair. Forty-five percent of the students are female and have long hair. Of the female students, 75% have long hair. Let F be the event that a student is female. Let L be the event that a student has long hair. One student is picked randomly. Are the events of being female and having long hair independent?

- The following probabilities are given in this example:
- $P(F) = 0.60$; $P(L) = 0.50$
- $P(F \text{ AND } L) = 0.45$
- $P(L|F) = 0.75$

NOTE

The choice you make depends on the information you have. You could use the first or last condition on the list for this example. You do not know $P(F|L)$ yet, so you cannot use the second condition.

Solution 1

Check whether $P(F \text{ AND } L) = P(F)P(L)$. We are given that $P(F \text{ AND } L) = 0.45$, but $P(F)P(L) = (0.60)(0.50) = 0.30$. The events of being female and having long hair are not independent because $P(F \text{ AND } L)$ does not equal $P(F)P(L)$.

Solution 2

Check whether $P(L|F)$ equals $P(L)$. We are given that $P(L|F) = 0.75$, but $P(L) = 0.50$; they are not equal. The events of being female and having long hair are not independent.

Interpretation of Results

The events of being female and having long hair are not independent; knowing that a student is female changes the probability that a student has long hair.

Try It Σ

3.12 Mark is deciding which route to take to work. His choices are I = the Interstate and F = Fifth Street.

- $P(I) = 0.44$ and $P(F) = 0.56$
- $P(I \text{ AND } F) = 0$ because Mark will take only one route to work.

What is the probability of $P(I \text{ OR } F)$?

Example 3.13

a. Toss one fair coin (the coin has two sides, H and T). The outcomes are ________. Count the outcomes. There are ____ outcomes.

b. Toss one fair, six-sided die (the die has 1, 2, 3, 4, 5 or 6 dots on a side). The outcomes are ________________. Count the outcomes. There are ___ outcomes.

c. Multiply the two numbers of outcomes. The answer is _______.

d. If you flip one fair coin and follow it with the toss of one fair, six-sided die, the answer in part c. is the number of outcomes (size of the sample space). What are the outcomes? (Hint: Two of the outcomes are $H1$ and $T6$.)

e. Event A = heads (H) on the coin followed by an even number (2, 4, 6) on the die. A = {_________________}. Find $P(A)$.

f. Event B = heads on the coin followed by a three on the die. B = {________}. Find $P(B)$.

g. Are A and B mutually exclusive? (Hint: What is $P(A \text{ AND } B)$? If $P(A \text{ AND } B) = 0$, then A and B are mutually exclusive.)

h. Are A and B independent? (Hint: Is $P(A \text{ AND } B) = P(A)P(B)$? If $P(A \text{ AND } B) = P(A)P(B)$, then A and B are independent. If not, then they are dependent).

Solution 3.13

a. H and T; 2

b. 1, 2, 3, 4, 5, 6; 6

c. 2(6) = 12

d. $T1, T2, T3, T4, T5, T6, H1, H2, H3, H4, H5, H6$

e. $A = \{H2, H4, H6\}$; $P(A) = \frac{3}{12}$

f. $B = \{H3\}$; $P(B) = \frac{1}{12}$

g. Yes, because $P(A \text{ AND } B) = 0$

h. $P(A \text{ AND } B) = 0.P(A)P(B) = \left(\frac{3}{12}\right)\left(\frac{1}{12}\right)$. $P(A \text{ AND } B)$ does not equal $P(A)P(B)$, so A and B are dependent.

Try It Σ

3.13 A box has two balls, one white and one red. We select one ball, put it back in the box, and select a second ball (sampling with replacement). Let T be the event of getting the white ball twice, F the event of picking the white ball first, S the event of picking the white ball in the second drawing.

a. Compute $P(T)$.

b. Compute $P(T|F)$.

c. Are T and F independent?.

d. Are F and S mutually exclusive?

e. Are F and S independent?

3.3 | Two Basic Rules of Probability

When calculating probability, there are two rules to consider when determining if two events are independent or dependent and if they are mutually exclusive or not.

The Multiplication Rule

If A and B are two events defined on a **sample space**, then: $P(A \text{ AND } B) = P(B)P(A|B)$.

This rule may also be written as: $P(A|B) = \dfrac{P(A \text{ AND } B)}{P(B)}$

(The probability of A given B equals the probability of A and B divided by the probability of B.)

If A and B are **independent**, then $P(A|B) = P(A)$. Then $P(A \text{ AND } B) = P(A|B)P(B)$ becomes $P(A \text{ AND } B) = P(A)P(B)$.

The Addition Rule

If A and B are defined on a sample space, then: $P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$.

If A and B are **mutually exclusive**, then $P(A \text{ AND } B) = 0$. Then $P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$ becomes $P(A \text{ OR } B) = P(A) + P(B)$.

Example 3.14

Klaus is trying to choose where to go on vacation. His two choices are: A = New Zealand and B = Alaska

- Klaus can only afford one vacation. The probability that he chooses A is $P(A) = 0.6$ and the probability that he chooses B is $P(B) = 0.35$.
- $P(A \text{ AND } B) = 0$ because Klaus can only afford to take one vacation
- Therefore, the probability that he chooses either New Zealand or Alaska is $P(A \text{ OR } B) = P(A) + P(B) = 0.6 + 0.35 = 0.95$. Note that the probability that he does not choose to go anywhere on vacation must be 0.05.

Example 3.15

Carlos plays college soccer. He makes a goal 65% of the time he shoots. Carlos is going to attempt two goals in a row in the next game. A = the event Carlos is successful on his first attempt. $P(A) = 0.65$. B = the event Carlos is successful on his second attempt. $P(B) = 0.65$. Carlos tends to shoot in streaks. The probability that he makes the second goal **GIVEN** that he made the first goal is 0.90.

a. What is the probability that he makes both goals?

Solution 3.15

a. The problem is asking you to find $P(A \text{ AND } B) = P(B \text{ AND } A)$. Since $P(B|A) = 0.90$: $P(B \text{ AND } A) = P(B|A) P(A) = (0.90)(0.65) = 0.585$

Carlos makes the first and second goals with probability 0.585.

b. What is the probability that Carlos makes either the first goal or the second goal?

Solution 3.15

b. The problem is asking you to find $P(A \text{ OR } B)$.

$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B) = 0.65 + 0.65 - 0.585 = 0.715$

Carlos makes either the first goal or the second goal with probability 0.715.

c. Are A and B independent?

Solution 3.15

c. No, they are not, because $P(B \text{ AND } A) = 0.585$.

$P(B)P(A) = (0.65)(0.65) = 0.423$

$0.423 \neq 0.585 = P(B \text{ AND } A)$

So, $P(B \text{ AND } A)$ is **not** equal to $P(B)P(A)$.

d. Are A and B mutually exclusive?

Solution 3.15

d. No, they are not because $P(A \text{ and } B) = 0.585$.

To be mutually exclusive, $P(A \text{ AND } B)$ must equal zero.

Try It Σ

3.15 Helen plays basketball. For free throws, she makes the shot 75% of the time. Helen must now attempt two free throws. C = the event that Helen makes the first shot. $P(C) = 0.75$. D = the event Helen makes the second shot. $P(D) = 0.75$. The probability that Helen makes the second free throw given that she made the first is 0.85. What is the probability that Helen makes both free throws?

Example 3.16

A community swim team has **150** members. **Seventy-five** of the members are advanced swimmers. **Forty-seven** of the members are intermediate swimmers. The remainder are novice swimmers. **Forty** of the advanced swimmers practice four times a week. **Thirty** of the intermediate swimmers practice four times a week. **Ten** of the novice swimmers practice four times a week. Suppose one member of the swim team is chosen randomly.

a. What is the probability that the member is a novice swimmer?

Solution 3.16

a. $\frac{28}{150}$

b. What is the probability that the member practices four times a week?

Solution 3.16

b. $\frac{80}{150}$

c. What is the probability that the member is an advanced swimmer and practices four times a week?

Solution 3.16

c. $\frac{40}{150}$

d. What is the probability that a member is an advanced swimmer and an intermediate swimmer? Are being an advanced swimmer and an intermediate swimmer mutually exclusive? Why or why not?

Solution 3.16

d. $P(\text{advanced AND intermediate}) = 0$, so these are mutually exclusive events. A swimmer cannot be an advanced swimmer and an intermediate swimmer at the same time.

e. Are being a novice swimmer and practicing four times a week independent events? Why or why not?

Solution 3.16

e. No, these are not independent events.
P(novice AND practices four times per week) = 0.0667
P(novice)P(practices four times per week) = 0.0996
0.0667 ≠ 0.0996

Try It Σ

3.16 A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is taking a gap year?

Example 3.17

Felicity attends Modesto JC in Modesto, CA. The probability that Felicity enrolls in a math class is 0.2 and the probability that she enrolls in a speech class is 0.65. The probability that she enrolls in a math class GIVEN that she enrolls in speech class is 0.25.

Let: M = math class, S = speech class, $M|S$ = math given speech

a. What is the probability that Felicity enrolls in math and speech?
Find $P(M \text{ AND } S) = P(M|S)P(S)$.

b. What is the probability that Felicity enrolls in math or speech classes?
Find $P(M \text{ OR } S) = P(M) + P(S) - P(M \text{ AND } S)$.

c. Are M and S independent? Is $P(M|S) = P(M)$?

d. Are M and S mutually exclusive? Is $P(M \text{ AND } S) = 0$?

Solution 3.17

a. 0.1625, b. 0.6875, c. No, d. No

Try It Σ

3.17 A student goes to the library. Let events B = the student checks out a book and D = the student check out a DVD. Suppose that $P(B) = 0.40$, $P(D) = 0.30$ and $P(D|B) = 0.5$.

a. Find $P(B \text{ AND } D)$.

b. Find $P(B \text{ OR } D)$.

Example 3.18

Studies show that about one woman in seven (approximately 14.3%) who live to be 90 will develop breast cancer. Suppose that of those women who develop breast cancer, a test is negative 2% of the time. Also suppose that in