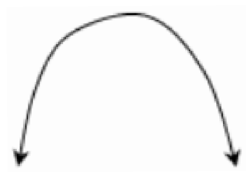
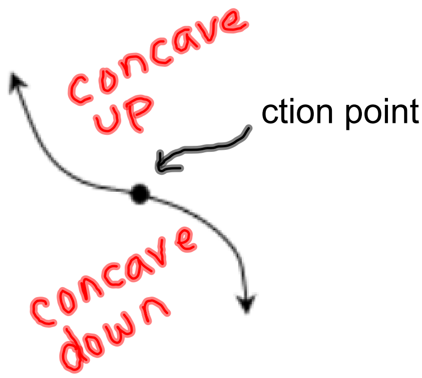
CONCAVITY AND
THE SECOND DERIVATIVE TEST

The first derivative describes the direction of the function. The second derivative describes the concavity of the original function. Concavity describes the direction of the curve, how it bends...

concave up



concave down



Just like direction, concavity of a curve can change, too. The points of change are called ***<u>inflection points</u>***.

<u>TEST FOR CONCAVITY</u>

If $f''(x) > 0$, then graph of f is concave up.
If $f''(x) < 0$, then graph of f is concave down.

EX #1: Given $f(x)=\frac{1}{3}x^3-x$, determine the open intervals on which the graph is concave upward or downward.

$f'(x) = x^2 - 1$

$x^2 - 1 = 0$

$x = 1, -1$

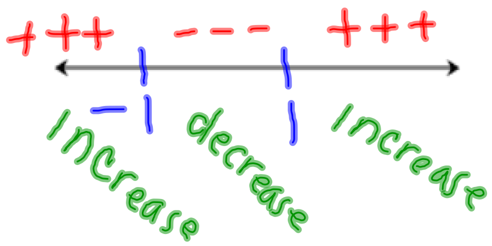
$f''(x) = 2x$

Since $f''(x)=0$ at:

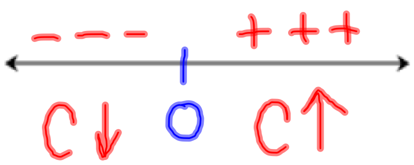
$2x = 0$

$x = 0$

Test Intervals:



$f'(x)$



$f''(x)$

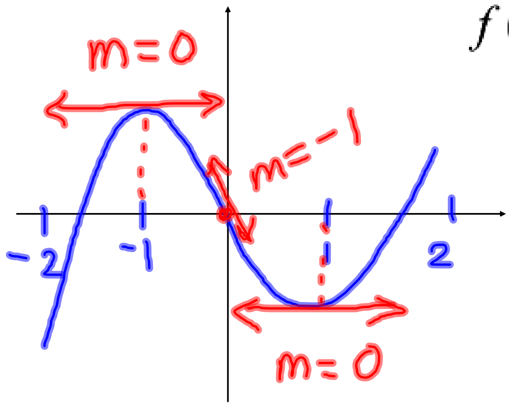
x	-1	0	1	
$f'(x)$	0	-1	0	
$f''(x)$	-2	0	2	

∴ concave down $(-\infty, 0)$

concave up $(0, \infty)$

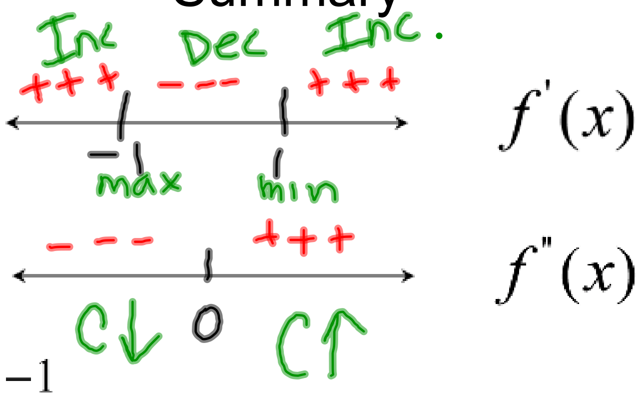
EX #2: Graphs and Derivatives

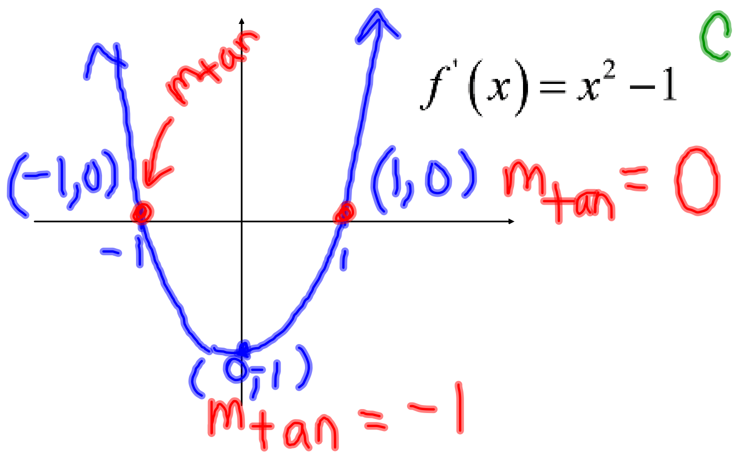
The concavity $\left(f''(x)\right)$ and direction $\left(f'(x)\right)$ of the function $\left(f(x)\right)$ is related to the slope of the derivative.



$$f(x)=\frac{1}{3}x^3-x$$

Summary



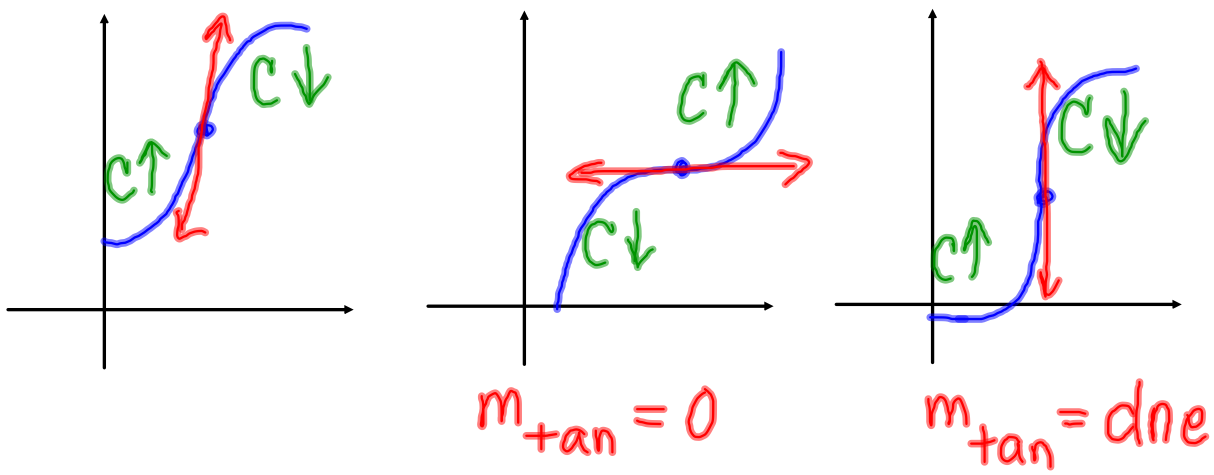


$$f'(x)=x^2-1$$

POINTS OF INFLECTION

The concavity of *f* changes at a point of inflection.

Where $f''(x) = 0$ or $f''(x) = \textit{does not exist}$



* The graph crosses its tangent line at a point of inflection.

EX #3: Determine points of inflection and discuss concavity of the graph of $f(x) = x^4 - 4x^3$

$f'(x) = 0$

$4x^3 - 12x^2 = 0$

$x = 0, 3$

$f''(x) = 0$

$12x^2 - 24x = 0$

$x = 0, 2$

$f'(x)$: $- - -$ | 0 | $- - -$ | 3 | $+ + +$

3: min

$f''(x)$: $+ + +$ | 0 | $- - -$ | 2 | $+ + +$

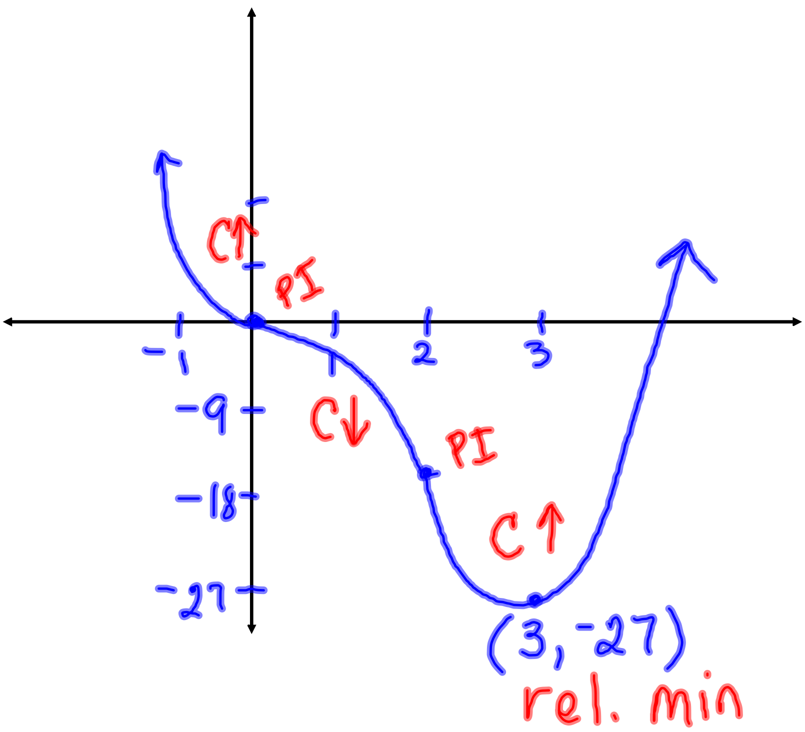
C↑ 0 C↓ 2 C↑

$f(0) = 0$

$f(2) = -16$

$f(3) = -27$

Sketch graph from above information:



EX #4: Use the Second Derivative Test to determine the relative extrema for $f(x) = -3x^5 + 5x^3$

Find critical numbers where $f'(x) = 0$

$-15x^4 + 15x^2 = 0$

$-15x^2(x^2 - 1) = 0$

$x = 0 \quad x = 1, -1$

Find $f''(x)$ $\quad -60x^3 + 30x = 0$

$-30x(2x^2 - 1) = 0$

$x = 0 \qquad x = \pm\sqrt{\frac{1}{2}}$

f'' $\quad -\frac{\sqrt{2}}{2} \qquad 0 \qquad \frac{\sqrt{2}}{2}$

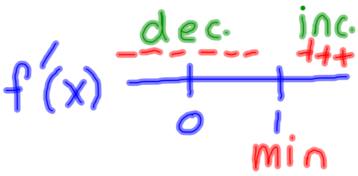
Point	(0,0)	(1,2)	(-1,-2)
Sign of $f''(x)$	$f''(0)=0$	$f''(1)<0$	$f''(-1)>0$
Conculsion	test fails	max	min

EX #5: Use First and Second Derivative Tests to determine behavior of *f* and graph.

Given: $f(x) = 3x^4 - 4x^3 + 6$

1. $f'(x) = 0$

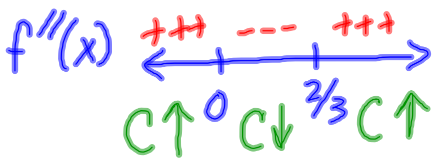
$12x^3 - 12x^2 = 0$
$12x^2(x-1) = 0$
$x = 0 \quad x = 1$



2. critical points $(0,6)$ and $(1,5)$

3. $f''(x) = 0$

$36x^2 - 24x = 0$
$12x(3x-2) = 0$
$x = 0 \quad x = 2/3$



4. Points of Inflection $(0,6)$ $(2/3, 5.4)$

5. Summarize

Critical Points (*c*)	*f* (*c*)	*f* '' (*c*)	Conculsion	Point of Inflection
0	6	0	none	(0,6)
1	5	12	concave up	(2/3, 5.4)

6. Graph