Intervals of Increasing & Decreasing

Find the first derivative of the function, f.

Find critical numbers.

Set the first derivative equal to zero and solve.

Determine whether the first derivative is undefined for any x-values.

Make sure these numbers are in the domain of the original function, f. If they are not, then the numbers are NOT critical points.

Plot these numbers on a number line and test the regions with the first derivative.

A positive result indicates the function is increasing on that interval.

A negative result indicates the function is decreasing on that interval.

Example 1: Determine intervals on which function $f(x) = x^4 - 4x^3 + 4x^2$ is <u>increasing or decreasing.</u>

$f'(x) = 4x^3 - 12x^2 + 8x$

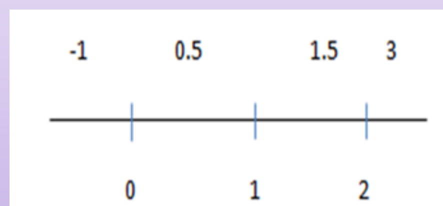
$f'(x) = 0$ ➔ $4x(x^2 - 3x + 2) = 0$

$4x(x-2)(x-1) = 0$ ➔ $x = 0, 1, 2$

The first derivative is defined for ALL x-values.

All critical numbers, namely x=0, 1, 2 are in the domain of the function f.

Plot the critical numbers on a number line and pick convenient numbers in those intervals. Start selecting numbers in the given intervals. Let's select a convenient number in the interval less than zero. How about -1? Then, select a number in the interval zero to 1, how about 0.5? Next, pick a number in the interval 1 to 2, let's say, 1.5. Finally select a number greater than 2, say, 3.



Test the regions with the <u>first derivative</u>:

$f'(-1) < 0$ ➔ f is decreasing on $(-\infty, 0)$

$f'\left(\frac{1}{2}\right) > 0$ ➔ f is increasing on $(0,1)$

$f'(1.5) < 0$ ➔ f is decreasing on $(1, 2)$

$f'(3) > 0$ ➔ f is increasing on $(2, \infty)$

$f'(3) > 0$ ➔ f is increasing on $(2, \infty)$

f is increasing on $(0,1) \cup (2, \infty)$

f is decreasing on $(-\infty, 0) \cup (1,2)$

Example 2: Determine the intervals on which the function $x\sqrt{x+3}$ is increasing or decreasing.

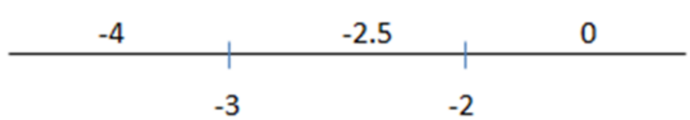
$$f'(x) = (x+3)^{\frac{1}{2}} + \frac{1}{2}x(x+3)^{-\frac{1}{2}}$$
$$= \frac{2(x+3)+x}{2\sqrt{x+3}}$$
$$= \frac{3x+6}{2\sqrt{x+3}}$$

$f'(x) = 0 \Rightarrow 3x + 6 = 0 \Rightarrow x = -2$

$f'(x)$ is undefined when $2\sqrt{x+3}$ =0. That is, $x = -3$.

Critical numbers $x = -3, -2$ are in the domain of function $f(x)$.

Plot the critical numbers on a number line and pick convenient numbers in those intervals. Start selecting numbers in the given intervals. Let's select a convenient number in the interval less than -3. How about -4? Then, select a number in the interval -3 to -2, how about -2.5? Finally pick a number in the interval greater than -2, say 0.



<u>Test the regions with the first derivative:</u>

$f'(-4)$ is not real

$f'(-2.5) < 0 \Rightarrow f$ is decreasing on $(-3, -2)$

$f'(0) > 0 \Rightarrow f$ is increasing on $(-2, \infty)$

f is increasing on $(-2, \infty)$

f is decreasing on $(-3, -2)$

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Graphing $f(x)$

Steps to graph $f(x)$

1. Set $f'(x)=0$ and solve for critical values.
2. Substitute the critical values into $f(x)$ to find critical points.
3. Set $f''(x)=0$ and solve for critical values.
4. Substitute the critical values into $f(x)$ to find inflection point(s).
5. Do the sign chart for $f'(x)$, to determine where $f(x)$ is increasing or decreasing
6. Do the sign chart for $f''(x)$, to determine where $f(x)$ is Concave up or down
7. Graph $f(x)$

$f'(x)$, $f''(x)$

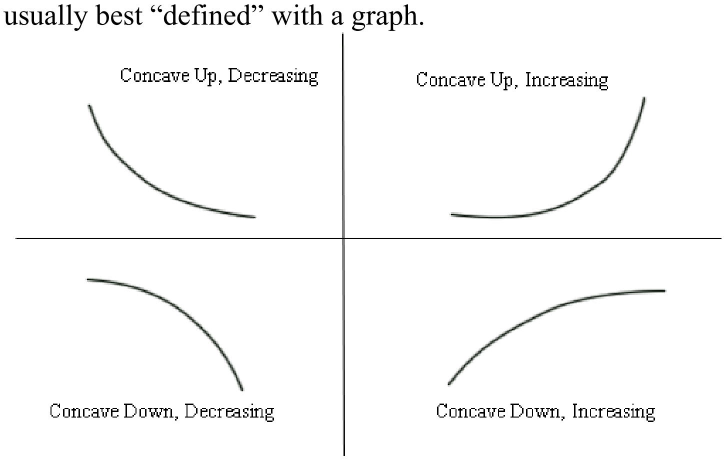
1) $f'(x)>0$, $f''(x)>0$ **Increasing at an increasing rate 2nd Quad**

2) $f'(x)>0$, $f''(x)<0$ **Increasing at a decreasing rate 4th Quad**

3) $f'(x)<0$, $f''(x)<0$ **Decreasing at an increasing rate 3rd Quad**

4) $f'(x)<0$, $f''(x)>0$ **Decreasing at decreasing rate 1st Quad**

Concavity: Concavity is usually best “defined” with a graph.



So a function is **concave up** if it opens up and the function is **concave down** if

Try to match the 1,2,3,4 with the above graphs

Example. Given $f(x) = x^2 - 8x + 10$

1. Find critical points and all x intervals the $f(x)$ is decreasing and increasing
2. Find inflection point all x intervals the $f(x)$ is concave up and down, also **graph** $f(x)$

$f'(x) = 2x - 8 = 0 \qquad x = 4$ Critical Values $\qquad f(4) = 16 - 32 + 10 = -6$ Critical Points

X	$-\infty$		4		∞
$f'(x) = 2x - 8$		$-$	0	$+$	
$y = f(x)$		Down ↘	-6	Up ↗	

$f''(x) = 2 > 0,$ Always Concave up

A. Given $f(x) = x^3 - 12x + 1$

1. Find critical points and all x intervals the $f(x)$ is decreasing and increasing
2. Find inflection point all x intervals the $f(x)$ is concave up and down, also **graph** $f(x)$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = (x-2)(x+2) = 0 \qquad x = -2,\ 2$ Critical Values

$f(x) = x^3 - 12x + 1 \qquad f(-2) = 17\ , \qquad f(2) = -15 \qquad (-2, 17) \qquad (2, -15)$ Critical Points

X	$-\infty$		-2		2		∞
$f'(x) = 3x^2 - 12$		$+$	0	$-$	0	$+$	
$y = f(x)$		↗	17	↘	-15	↗	

$f(x) = x^3 - 12x + 1 \qquad f'(x) = 3x^2 - 12 \qquad f''(x) = 6x = 0 \qquad x = 0 \qquad f(0) = 1$ Inflection Point

X	$-\infty$		0		∞
$f''(x) = 6x$		$-$	0	$+$	
$y = f(x)$		↘ Down	1	Up ↗	

B. Given $f(x) = x^4 - 4x^3$

1. Find critical points and all x intervals the $f(x)$ is decreasing and increasing
2. Find inflection point all x intervals the $f(x)$ is concave up and down, also **graph** $f(x)$

C. Given $f(x) = 10 + 27x - x^3$, on the interval $0 \le x \le 4$

1. Find critical points and all x intervals the $f(x)$ is decreasing and increasing
2. Find inflection point all x intervals the $f(x)$ is concave up and down, also **graph** $f(x)$
3. Find absolute min and max in the given interval

D. Given $f(x) = x^3 + 3x^2 - 1$, on the interval $-3 \le x \le 1$

1. Find critical points and all x intervals the $f(x)$ is decreasing and increasing
2. Find inflection point all x intervals the $f(x)$ is concave up and down, also **graph** $f(x)$
3. Find absolute min and max in the given interval

E. For what value(s) of x will the function $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x$, on the interval $-10 \le x \le 10$ be largest?

F: Find 2 positive numbers whose product is 25 and whose sum is a minimum.

G What is the maximum value of $f(x) = x^4 - 8x^2 - 3$, on the interval $-3 \le x \le 3$?

H A shepherd wishes to build a rectangular fenced area against the side of a barn. He has 360 feet of fencing material, and only needs to use it on three sides of the enclosure, since the wall of the barn will provide the last side. What dimensions should the shepherd choose to maximize the area of the enclosure?

I. Adam has 3000 yd of fencing material with which to enclose a rectangular portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is this Area?

Solution

E. $f(x) = x^2 - 4x - 5 = (x-5)(x+1) = 0$ so $f'(x) = 0$ at $x = 5$ and $x = -1$.

Now, to classify these points, the second derivative test can be used.

$f''(x) = 2x - 4$;
$f''(5) = 6$, which is greater than zero, so $f(5)$ is a local minimum.
$f''(-1) = -6$,which is less than zero, so $f(-1)$ is a local maximum.

Finally, the actual values of the local maxima and the endpoints should be compared to each other to see which of these the absolute maximum is.

$f(-10) = -283\frac{1}{3}$
$f(-1) = 2\frac{2}{3}$
$f(10) = 83\frac{1}{3}$

So, the absolute maximum on the interval occurs when $x = 10$.

F. Objective: $S = x + y$. The goal is to minimize S. Constraint: $xy = 25$. Substitute constraint into objective:

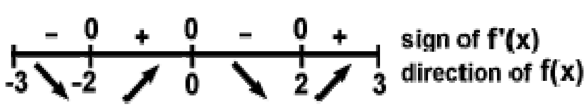
$$y = 25/x \qquad S = x + \frac{25}{x} \quad 0 < x < \infty \quad \Rightarrow \quad S' = 1 + \frac{-25}{x^2} \quad \Rightarrow \quad S' = 1 + \frac{-25}{x^2} = 0 \quad \Rightarrow \quad x = 5$$

Use second derivative to classify: $S'' = \dfrac{50}{x^2} \quad \Rightarrow \quad S''(5) = \dfrac{50}{5^2} = 2 > 0$

$S''(5) > 0$, so S has a local min at $x = 5$. However, notice that $S''(x)$ is always positive on the interval $(0,\infty)$, so S is always concave up on that interval, which means that the local min is also the absolute min. Therefore, 5 and 5 are the positive numbers with the smallest sum whose product is 25.

G. $f(x) = 4x^3 - 16x = 4x(x^2 - 4) = 0$

This equals zero when x equals 0 or -2 or +2. Making use of the sign of the first derivative yields the following chart for the behavior of f:



Based on this, the only local maximum occurs at $x = 0$. Now, to see if this is an absolute maximum on the interval, compare the value of f at $x = 0$ to the value at the endpoints. $f(-3) = 6$
$f(0) = -3$
$f(3) = 6$
While $x = 0$ is a local maximum, it is not the absolute maximum on this interval. The absolute maximum occurs at both of the endpoints in this case.

H. Below is a sketch of the situation:

Objective: maximize $A = xy$.
Constraint: $2y + x = 360$.
Substitution into objective: $A = (360 - 2y)(y)$
$A(y) = 360y - 2y^2$
$A'(y) = 360 - 4y$
$A'(y) = 0$ at $x = 90$
$A''(y) = -4$, so $A''(90) < 0$ and $y = 90$ is a local maximum. However, because $A''(y) = -4$ for all y, the graph of $A(y)$ is always concave down, so the local maximum is also the absolute maximum. Thus, choosing $y = 90$ft and $x = 180$ ft will generate the largest area.

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