Optimization- What is the Minimum or Maximum?

Here is an application of calculus (finally…) that is utilized by many in their daily lives. Say, I have some amount of fencing and I want to find out the dimensions that would give me the largest area? Its Christmas time and I have to make a gift box. I have some paper that I could fold to make the box. Again, what dimensions do I use to maximize volume? As a business manager one often asks questions about how one can minimize costs?

The answers to all these questions lie in *Optimization*.

Here are a few steps to solve optimization problems:

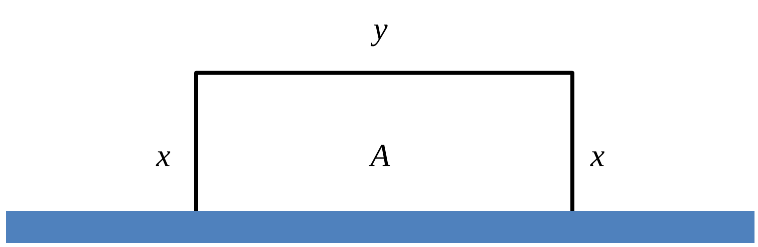
1. Read the problem- write the knowns, unknowns, and draw a diagram if applicable.
2. Write down an equation for what needs to be maximized/minimized (such as A=b*h or Cost= (price)*(number of units) etc.)
3. Write the function in step 2 in terms of one variable by using a giving relationship from step 1 i.e. write A= in terms of base only or cost = f(price) only (this step become clear in the following example)
4. Now find the derivative of the function found in step 3
5. Find the critical values (i.e. set the derivative = 0 and solve for the variable)
6. The critical value obtained is either the maximum or minimum (check for yourself on the number line). Find the other variable.

In order to understand this further, let's solve a sample problem.

Q. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

1. *Read the problem- write the knowns, unknowns, and draw a diagram if applicable*

Perimeter of fencing = 2400
Area = ?



2. *Write down an equation for what needs to be maximized/minimized*

A= base*height = xy

Optimization- What is the Minimum or Maximum?

3. *Write the function in step 2 terms of one variable by using a giving relationship from step___*

We know that the perimeter of fence = 2400. In our case that means $2x + y = 2400$. This tells us

$$y = 2400-2x$$

Therefore area can be written as $A= x(2400-2x) = 2400x -2x^2$

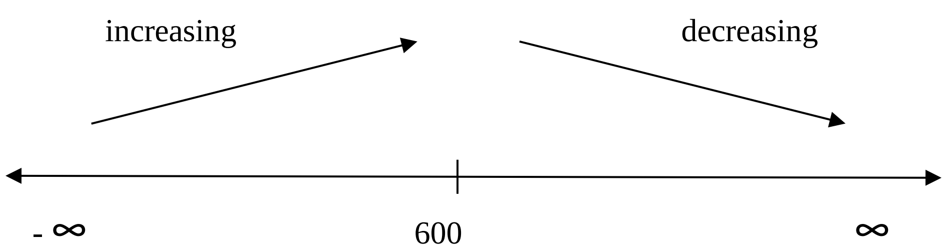
4. *Now find the derivative of the function found in step 3*

$$A' = 2400-4x$$

5. *Find the derivate of the function found in step 3*

$$2400 – 4x = 0$$
$$x= 600$$

6. *The critical value obtained is either the maximum or minimum. Find the other variable.*



$A'(x) = 2400-4x$
$A' (500) = 2400-4(500) = 400$
$A' (700) = 2400- 4(700) = -400$

If width (x) = 600 feet,

then length (y) = 2400- 2x = 2400 – 1200 = 1200 feet

Thus the rectangular field should be ***600 feet*** *wide and* ***1200 feet*** *long.*

Optimization Problems

1) A farmer has 400 yards of fencing and wishes to fence three sides of a rectangular field (the fourth side is along an existing stone wall, and needs no additional fencing). Find the dimensions of the rectangular field of largest area that can be fenced.

$2x + y = 400 \Rightarrow y = 400 - 2x$

$A(x) = x(400 - 2x) = 400x - 2x^2$

$A'(x) = 400 - 4x \quad 400 - 4x = 0 \Rightarrow x = 100$

$A''(x) = -4$

By the 2nd derivative test, the dimensions would be 100 yd by 200 yd.

3) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$5 per meter, and the fencing for the other sides costs \$3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose.

Call the length of fence along the river *x*, and the length perpendicular to the river *y*.

$C(x) = 5x + 3(2y + x) \qquad xy = 1200 \Rightarrow y = \frac{1200}{x} \Rightarrow C(x) = 8x + \frac{7200}{x}$

$C'(x) = 8 - \frac{7200}{x^2} \quad 8 - \frac{7200}{x^2} = 0 \Rightarrow 8x^2 = 7200 \Rightarrow x^2 = 900 \Rightarrow x = 30$

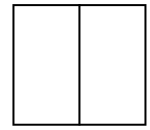
$C''(x) = \frac{14400}{x^3} \quad C''(30) = \frac{14400}{30^3} > 0$

By the 2nd derivative test, a field that is 30 m along the river by 40 m perpendicular to the river would be least expensive.

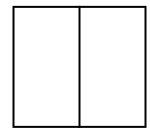
Pre-Calculus
Optimization Problems

Fencing Problems

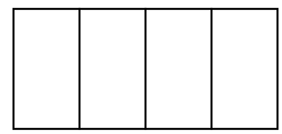
1. A farmer has 480 meters of fencing with which to build two animal pens with a common side as shown in the diagram. Find the dimensions of the field with the maximum area. What is the maximum area?



2. Max plans to build two side-by-side identical rectangular pens for his pigs that will enclose a total area of 216 square feet. What is the minimum length of fencing he will need? What are the dimensions of the total enclosure?

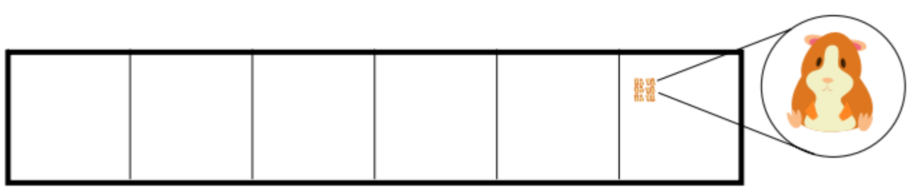


3. Build a rectangular pen with three parallel partitions using 500 feet of fencing. What overall dimensions will maximize the total area of the pen? What is the maximum area?

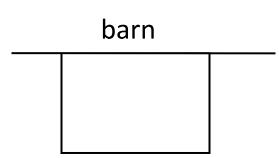


4. You are to fence a rectangular area. The fencing for the left and right sides costs $20 per foot and the fencing for the front and back sides costs $30 per foot. Find the dimensions of the rectangular area that result in the least cost, if the area inside the fencing is to be 3200 square feet. What is the cost? What are the dimensions?

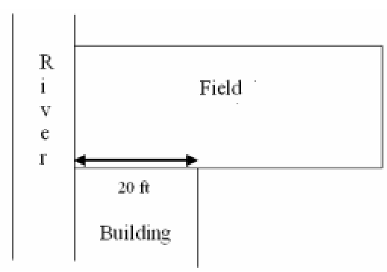
5. Your dream of becoming a hamster breeder has finally come true. You are constructing a set of rectangular pens in which to breed your furry friends. The overall area you are working with is 60 square feet, and you want to divide the area up into six pens of equal size as shown below. The cost of the outside fencing is $10 a foot. The interior fencing costs $5 a foot. You wish to minimize the cost of the fencing. Find the minimum cost and the overall dimensions of the enclosure.



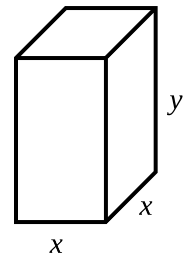
6. A rectangular lettuce patch, 480 square feet in area, is to be fenced off against rabbits. Find the least amount of fencing required if one side of the land is already protected by a barn. What are the dimensions?



7. A rectangular field as shown is to be bounded by a fence. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fencing. You can assume that fencing is not required along the river and the building. What is the max area?



Boxes (Rectangular Prisms)

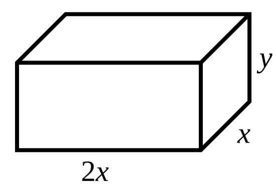


1. An open-top rectangular box with square base is to be made from 200 square feet of material. What dimensions will result in a box with the largest possible volume? What is the volume?

2. An open-top rectangular box with square base is to be made from 1200 square cm of material. What dimensions will result in a box with the largest possible volume? What is the volume?

3. A closed-top rectangular container with a square base is to have a volume 300 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost. What is that cost?

4. An open-top box will be constructed with material costing \$7 per square meter for the sides and \$13 per square meter for the bottom. The dimensions of the bottom are to have its length equal to twice its width (see diagram). Find the dimensions of the box of largest volume than can be built with at most \$300 of materials. What is the volume?



5. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box. What is that cost?

6. A pool with a square bottom is to have a volume of 2000 cubic feet. The owners plan to use a fancy tile to complete the pool. The sides of the pool will cost \$80 per square foot and the bottom of the pool will cost \$40 per square foot. Find the pool dimensions that will minimize the cost of construction.

7. Find the dimensions of the least expensive rectangular box which is three times as long as it is wide and which holds 100 cubic centimeters of water. The material for the bottom costs 7¢ per cm^2, the sides cost 5¢ per cm^2 and the top cost 2¢ per cm^2. Also find the cost.

Answers:

Fencing:

1. overall dimensions: 80m x 120m; Maximum Area = 9600m^2
2. overall dimensions: 12 feet x 18 feet; Minimum Fencing: 72 feet
3. overall dimensions: 50 feet x 125 feet; Maximum Area = 6,250 ft^2
4. overall dimensions: 46.188 x 69.282; Minimum Cost = $5542.56
5. overall dimensions: 5.164 feet x 11.619; Minimum Cost = $464.76
6. dimensions: 15.492 x 30.984; Minimum Fencing: 61.968 ft
7. dimensions: 255 feet x 510 feet; Maximum Area = 130,050 ft^2

Boxes:

1. x = 8.165 ft. and y = 4.082 ft, Volume = 272.166 ft^3
2. V = 4000 cm^3; box dimensions are 20 x 20 x 10 cm.
3. The cost is minimized when the dimensions are 9.655x9.655x3.218. The cost is $1118.60.
4. Dimensions: 1.96 x 3.92 x 2.43 m; Maximum Volume = 18.68 m^3.
5. Dimensions: 1.882 x 5.646 x 4.705 ft; Minimum Cost = $637.60
6. Dimensions are 20 x 20 x 5 feet; Minimum Cost is $48,000.
7. Dimensions are 2.912 x 8.736 x 3.931 cm, Minimum Cost = $6.87

Cylinders:

1. r = 2.879 cm, h = 5.759 cm, Minimum Surface Area = 156.282 cm^2
2. r = 3.742 ft, h = 3.742 ft, Maximum Volume = 164.567 ft^3
3. r = 7.663 in, h = 5.109 in , Minimum Cost = $2213.81
4. r = 4.924 cm, h = 16.412 cm, Minimum Cost = $22.85
5. r = 3.700 in, h = 23.249 in, Minimum Cost = $55.78
6. r = 5.150 cm, h = 10.301 cm, Maximum Volume = 858.387 cm^3
7. r = 4 ft, h = 2 ft, the paper is 4×2. Maximum Volume = 100.53 ft^3
8. r = 2.667 cm, h = 2.667 cm, Maximum Volume = 59.574 cm^3

Distances:

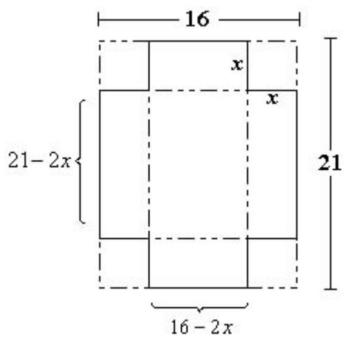
1. point: (2, 3) Minimum Distance: 6.708
2. point: (3.5, 1.871) Minimum Distance: 1.936
3. point: (1.719, 1.046) Minimum Distance: 3.417
4. Point: (0.824, 0.559) Minimum Distance: 3.538
5. Point C to Stake: 4 meters; Minimum Wire Length: 14.142 m
6. x = 5.714 m; Minimum Wire Length: 29 m

Cables & others…:

1. x = 80 m; Minimum Cost = $4900
2. x = 0.353 m; Minimum Cost = $7828.42
3. x = 1899.162 ft; Minimum Cost = $169,497.71
4. x = 7.817miles; Minimum Cost = $63,309.14
5. x = 3 miles; Shortest Time = 3.47 hours
6. x = 537 ft; Stay on the sidewalk for 1463 ft. Shortest time: 445 seconds (7.4 minutes)

Math 105a **Solutions**
Quiz 16

An open-top box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 inches wide and 21 inches long by cutting the same size square from each corner and then bending up the resulting sides. Find the size of the corner square (to be removed) that will produce a box having the largest possible volume. You may disregard the thickness of the cardboard.



Goal: Maximize the volume of the box.

The volume of the box is $V = lwh$, where length $l = 21 - 2x$, width $w = 16 - 2x$, and height $h = x$.
Therefore, $V(x) = (21 - 2x)(16 - 2x)x = 4x^3 - 74x^2 + 336x$

so $V'(x) = 12x^2 - 148x + 336 = 4(x - 3)(3x - 28)$.

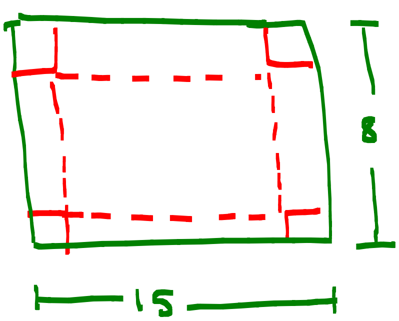
Setting $V'(x) = 0$ gives critical points $x = 3$ and $x = 28/3$. However, given the context of the problem $x = 28/3$ is not appropriate because it would lead to a negative width for the box since $w = 16-2\left(\frac{28}{3}\right) < 0$. Therefore, we can eliminate $x = 28/3$ as a possible location for the maximum to occur.

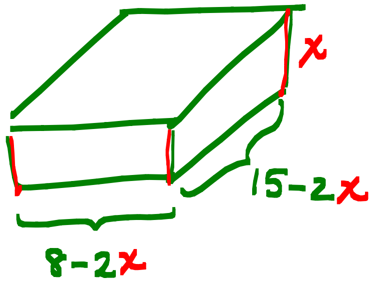
We can use the Second Derivative Test to verify that V has a local maximum at $x = 3$.

$$V''(x) = 24x - 148 \quad \Longrightarrow \quad V''(3) < 0$$

Therefore, the volume of the box is maximized when a 3 inch × 3 inch square is cut from the corners of the cardboard sheet.

3. You are planning to make an open rectangular box from an 8- by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume that you can create this way, and what is its volume?





$8-2x>0 \Rightarrow 0<x<4$

$V(x)=(8-2x)(15-2x)x=4x^3-46x^2+120x$

$V'(x)=12x^2-92x+120=0$

$3x^2-23x+30=0$

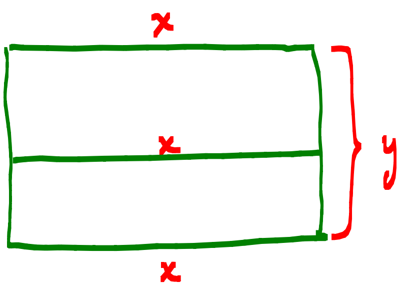
$(3x-5)(x-6)=0$

$x=5/3 \quad \cancel{x=6}$

Dimensions of the box: $\frac{5}{3}$ in. x $\frac{14}{3}$ in. x $\frac{35}{3}$ in.

Max. Volume: $\frac{2,450}{27}$ in.^{3}

4. A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?



$216=xy$

$F(x,y)=3x+2y \qquad F(x)=3x+\frac{432}{x}$

$F'(x)=3-\frac{432}{x^2}=\frac{3x^2-432}{x^2}=0$

$\Rightarrow x^2=144 \Rightarrow x=12 \Rightarrow y=18$

The dimensions are: 12m x 18m

Amount of fence: 72m