Abe Mirza $\lim_{x\to a} f(x) = L$ **Math**

We are interested to know what happen to function $f(x)$ as x approaches a number such as $x = a$ ($x \to a$).

Do we get a value such as L, or we do not get a value and call it as limit does not exist (DNE) as $x \to a$.

Mathematically we can express the expression of limit as $\lim_{x\to a} f(x) = L$ **or** $\lim_{x\to a} f(x) = DNE$

One-sided limit.

Obviously we can approach $x \to a$ from two sides, and see what happens;

negative side $x \to a^-$ so limit expression will then be $\lim_{x\to a^-} f(x)$

or

positive side $x \to a^+$ so limit expression will then be $\lim_{x\to a^+} f(x)$

How to do a limit problem?

1. Evaluate the expression with $x = a$ (disregard negative or positive side), if you get a value then this is the limit value.

Example 1: $\lim_{x\to 3^-} x^2 + 4 = 3^2 + 4 = 13$ **,** $\lim_{x\to 3^+} x^2 + 4 = 3^2 + 4 = 13$

Example 2: $\lim_{x\to -2^+} x(3 - \sqrt{2+x}) = -2(3 - \sqrt{2-2}) = -6$

(But be careful)

Example 3: $\lim_{x\to -2^-} x(3 - \sqrt{2+x}) =$ No limit, because if you approach $x = a$ from negative side inside the $\sqrt{}$ will be negative and there is no real answer.

$$\lim_{x\to -2^-} x(3 - \sqrt{2 - 2.001}) = -2.001(3 - \sqrt{-.001})$$

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2. If you do not get a value for its limit (indeterminate form $0/0$) then try to **simplify** the expression and apply the limit.

Example 4: $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4} = ?$ **,** $\dfrac{16 - 16}{4 - 4} = \dfrac{0}{0}$ **,**

let's simplify $\dfrac{x^2 - 16}{x - 4} = \dfrac{(x-4)(x+4)}{x-4} = (x+4)$**, now** $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4} = \lim_{x\to 4} (x+4) = 8$

Is $f(x)$ **continuous at point** $x = a$ **? Yes, if** $\lim_{x\to a} f(x) = f(a)$

It means that the limit as $x \to a$ exists and has the same value as of $f(a)$.

$$\lim_{x\to -\infty} f(x) = L \quad \textbf{or} \quad \lim_{x\to \infty} f(x) = L$$

1. If it is **not** a rational expression, simply replace x by $(-\infty$ or $\infty)$

Example 5: $\lim_{x\to-\infty} x^2+4 = (-\infty)^2+4 = \infty+4 = \infty$,

Example 6: $\lim_{x\to\infty} x^2+4 = \infty^2+4 = \infty+4 = \infty$,

Example 7: $\lim_{x\to-\infty} x^3+4 = (-\infty)^3+4 = -\infty+4 = -\infty$,

Example 8: $\lim_{x\to\infty} x^3+4 = (\infty)^3+4 = \infty+4 = \infty$,

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2. If it is a rational expression, divide every term by the highest exponent of the variable in the denominator and then apply the limit as x approaches to $(-\infty$ or $\infty)$

Example 9: $\lim_{x\to\infty} \frac{x^2+4}{x} = \lim_{x\to\infty} \frac{x^2/x+4/x}{x/x} = \lim_{x\to\infty} \frac{x+4/x}{1} = \lim_{x\to\infty} \frac{\infty+4/\infty}{1} = \lim_{x\to\infty} \frac{\infty+0}{1} = \infty$,

Example 10: $\lim_{x\to\infty} \frac{x}{x^2+4} = \lim_{x\to\infty} \frac{x/x^2}{x^2/x^2+4/x^2} = \lim_{x\to\infty} \frac{1/x}{1+4/x^2} = \lim_{x\to\infty} \frac{1/\infty}{1+4/\infty^2} = \lim_{x\to\infty} \frac{0}{1+0} = 0$,

Example 9: $\lim_{x\to\infty} \frac{5x^2+4}{2x^2-3x} = \lim_{x\to\infty} \frac{5x^2/x^2+4/x^2}{2x^2/x^2-3x/x^2} = \lim_{x\to\infty} \frac{5+4/x^2}{2-3/x} = \lim_{x\to\infty} \frac{5+4/\infty^2}{2-3/\infty} = \lim_{x\to\infty} \frac{5+0}{2-0} = \frac{5}{2}$,

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Composite functions

Given, $f(x)$, and $g(x)$ how to find the following composite functions?

1. $f(f(x))$ **2.** $g(g(x))$ **3.** $f(g(x))$ **4.** $g(f(x))$

We work from inside out.

Example 1. Let's $f(x) = x+1$, and $g(x) = x^2$

$f(f(x)) = f(x+1) = (x+1)+1 = x+2$

$g(g(x)) = g(x^2) = (x^2)^2 = x^4$

$f(g(x)) = f(x^2) = x^2+1$

$g(f(x)) = g(x+1) = (x+1)^2$

Example 2. Let's $f(x) = x-3$, and $g(x) = (x-1)^2$, find

$f(f(5)) = f(5-3) = f(2) = 2-3 = -1$

$g(g(5)) = g(4^2) = g(16) = 16^2 = 256$

$f(g(5)) = f(4^2) = f(16) = 16-3 = 13$

$g(f(5)) = g(5-3) = g(2) = (1)^2 = 1$

Practice problems

Find the limits for the following problems;

1. $\lim_{x\to 1^-} x+3$ **2.** $\lim_{x\to 1^+} x+3$ **3.** $\lim_{x\to 1} x+3$ **4.** Is $f(x)=x+3$ continuous at $x=1$**?**

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5. $\lim_{x\to 1^-} \frac{1}{x-1}$ **6.** $\lim_{x\to 1^+} \frac{1}{x-1}$ **7.** $\lim_{x\to 1} \frac{1}{x-1}$ **8.** Is $f(x)=\frac{1}{x-1}$ continuous at $x=1$**?**

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9. $\lim_{x\to 1^-} \frac{x^2-1}{x-1}$ **10.** $\lim_{x\to 1^+} \frac{x^2-1}{x-1}$ **11.** $\lim_{x\to 1} \frac{x^2-1}{x-1}$ **12.** Is $f(x)=\frac{x^2-1}{x-1}$ continuous at $x=1$**?**

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13. $\lim_{x\to -1^-} \frac{x(x+1)}{x+1}$ **14.** $\lim_{x\to -1^+} \frac{x(x+1)}{x+1}$ **15.** $\lim_{x\to -1} \frac{x(x+1)}{x+1}$ **16.** Is $f(x)=\frac{x(x+1)}{x+1}$ continuous at $x=-1$**?**

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17. $\lim_{x\to 0^-} \frac{x^2+2x}{x}$ **18.** $\lim_{x\to 0^+} \frac{x^2+2x}{x}$ **19.** $\lim_{x\to 0} \frac{x^2+2x}{x}$ **20.** Is $f(x)=\frac{x^2+2x}{x}$ continuous at $x=0$ **?**

Given $f(x)=\begin{cases} x-2 & if\ x\le 1 \\ x+2 & if\ x>1 \end{cases}$

21. $\lim_{x\to 1^-} f(x)$ **22.** $\lim_{x\to 1^+} f(x)$ **23.** $\lim_{x\to 1} f(x)$ **24.** Is $f(x)$ continuous at $x=1$**?**

Given $f(x)=\begin{cases} x^2-1 & if\ x<2 \\ x+1 & if\ x\ge 2 \end{cases}$

25. $\lim_{x\to 2^-} f(x)$ **26.** $\lim_{x\to 2^+} f(x)$ **27.** $\lim_{x\to 2} f(x)$ **28.** Is $f(x)$ continuous at $x=2$ **?**

Given $f(x)=\begin{cases} \sqrt{x-5} & if\ x\ge 9 \\ -1+\sqrt{x} & if\ x<9 \end{cases}$

29. $\lim_{x\to 9^-} f(x)$ **30.** $\lim_{x\to 9^+} f(x)$ **31.** $\lim_{x\to 9} f(x)$ **32.** Is $f(x)$ continuous at $x=9$ **?**

33. $\lim_{x\to-\infty}\frac{x^2+2x}{3x+1}$ **34.** $\lim_{x\to\infty}\frac{x^2+2x}{3x+1}$ **35.** $\lim_{x\to-\infty}\frac{3x+1}{x^2+2x}$ **36.** $\lim_{x\to\infty}\frac{3x+1}{x^2+2x}$

37. $\lim_{x\to-\infty}\frac{3x^2+2x}{x^2+1}$ **38.** $\lim_{x\to\infty}\frac{3x^2+2x}{x^2+1}$ **39.** $\lim_{x\to-\infty}\frac{x^2+1}{3x^2+2x}$ **40.** $\lim_{x\to\infty}\frac{x^2+1}{3x^2+2x}$

Answers

1	**2**	**3**	**4**		**5**	**6**	**7**	**8**
4	4	4	yes		$-\infty$	∞	DNE	no
9	**10**	**11**	**12**		**13**	**14**	**15**	**16**
2	2	2	no		–1	–1	–1	no
17	**18**	**19**	**20**		**21**	**22**	**23**	**24**
2	2	2	no		–1	3	DNE	no
25	**26**	**27**	**28**		**29**	**30**	**31**	**32**
3	3	3	yes		2	2	2	yes
33	**34**	**35**	**36**		**37**	**38**	**39**	**40**
$-\infty$	∞	0	0		3	3	$1/3$	$1/3$

Composite functions

Given $f(x)=2x-1$, and $g(x)=(x+1)^2$, find

1. $f(f(x))$ **2.** $g(g(x))$ **3.** $f(g(x))$ **4.** $g(f(x))$

5. $f(f(-3))$ **6** $g(g(-3))$ **7.** $f(g(-3))$ **8.** $g(f(-3))$

Given $f(x)=x-3$, and $g(x)=\sqrt{x+5}$, find

9. $f(f(x))$ **10.** $g(g(x))$ **11.** $f(g(x))$ **12.** $g(f(x))$

13. $f(f(4))$ **14** $g(g(4))$ **15.** $f(g(4))$ **16.** $g(f(4))$

1	**2**	**3**	**4**		**5**	**6**	**7**	**8**
$4x-3$	$\left((x+1)^2+1\right)^2$	$2(x+1)^2-1$	$(2x)^2$		-15	25	7	36
9	**10**	**11**	**12**		**13**	**14**	**15**	**16**
$x-6$	$\sqrt{\sqrt{x+5}+5}$	$\sqrt{x+5}-3$	$\sqrt{x+2}$		–2	$\sqrt{8}$	0	$\sqrt{6}$