11.3 The Hyperbola and Functions Defined by Radials

Objectives

1 Recognize the equation of a hyperbola.

2 Graph hyperbolas by using asymptotes.

3 Identify conic sections by their equations.

4 Graph certain square root functions.

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Recognize the equation of a hyperbola.

A **hyperbola** is the set of all points in a plane such that the absolute value of the ***difference*** of the distances from two fixed points (called ***foci***) is constant.

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Recognize the equation of a hyperbola.

Equations of Hyperbolas

A hyperbola with *x*-intercepts (*a*, 0) and (–*a*, 0) has an equation of the form.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

A hyperbola with *y*-intercepts (0, *b*) and (0, –*b*) has an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

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Graph hyperbolas by using asymptotes.

Asymptotes of Hyperbolas

The extended diagonals of the rectangle with vertices (corners) at the points (*a*, *b*), (–*a*, *b*), (–*a*, –*b*), and (*a*, –*b*) are the **asymptotes** of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This rectangle is called the **fundamental rectangle.** The equations of the asymptotes are:

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

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Graph hyperbolas by using asymptotes.

Graphing a Hyperbola

Step 1 **Find the intercepts**. Locate the intercepts at (*a*, 0) and (–*a*, 0) if the x^2-term has a positive coefficient, or at (0, *b*) and (0, –*b*) if the y^2-term has a positive coefficient.

Step 2 **Find the fundamental rectangle.** Locate the vertices of the fundamental rectangle at (*a*, *b*), (–*a*, *b*), (–*a*, –*b*), and (*a*, –*b*).

Step 3 **Sketch the asymptotes.** The extended diagonals of the rectangle are the asymptotes of the hyperbola, and they have equations

$$y = \pm\frac{b}{a}x.$$

Step 4 **Draw the graph.** Sketch each branch of the hyperbola through an intercept and approaching (but not touching) the asymptotes.

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CLASSROOM EXAMPLE 1 **Graphing a Horizontal Hyperbola**

Graph $\frac{x^2}{81} - \frac{y^2}{64} = 1.$

Solution:

Step 1 *a* = 9 and *b* = 8
x-intercepts at (9, 0) and (–9, 0)

Step 2 The four points and vertices of the fundamental rectangle are:

(*a*, *b*) = (9, 8) (–*a*, *b*) = (–9, 8)
(–*a*, –*b*) = (–9, –8) (*a*, –*b*) = (9, –8)

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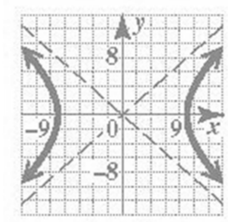
CLASSROOM EXAMPLE 1 — Graphing a Horizontal Hyperbola (cont'd)

Step 3 The equations of the asymptotes are:
x-intercepts at (9, 0) and (–9, 0)

$$\frac{x^2}{81} - \frac{y^2}{64} = 1$$

$$y = \pm\frac{8}{9}x$$

(9, 8)
(–9, 8)
(–9, –8)
(9, –8)



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CLASSROOM EXAMPLE 2 — Graphing a Vertical Hyperbola

Graph $\frac{y^2}{9} - \frac{x^2}{25} = 1.$

Solution:

Step 1 a = 5 and b = 3
y-intercepts at (0, 3) and (0, –3)

Step 2 The four points and vertices of the fundamental rectangle are:

(5, 3) (–5, 3) (–5, –3) (5, –3)

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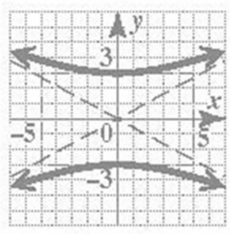
CLASSROOM EXAMPLE 2 — Graphing a Vertical Hyperbola (cont'd)

Step 3 The equations of the asymptotes are:
y-intercepts at (0, 3) and (0, –3)

$$\frac{y^2}{4} - \frac{x^2}{25} = 1$$

$$y = \pm\frac{2}{5}x$$

(5, 3)
(–5, 3)
(–5, –3)
(5, –3)



As with circles and ellipses, hyperbolas are graphed with a graphing calculator by first writing the equations of two functions whose union is equivalent to the equation of the hyperbola. A square window gives a truer shape for hyperbolas, too.

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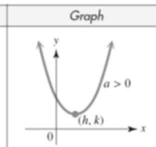
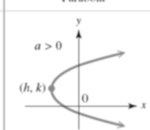
Graph hyperbolas by using asymptotes.

Summary of Conic Sections

Equation	Graph	Description	Identification
$y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$	 Parabola	It opens up if $a > 0$, down if $a < 0$. The vertex is (h, k).	It has an x^2-term. y is not squared.
$x = ay^2 + by + c$ or $x = a(y - k)^2 + h$	 Parabola	It opens to the right if $a > 0$, to the left if $a < 0$. The vertex is (h, k).	It has a y^2-term. x is not squared.

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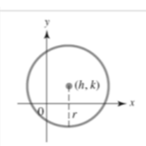
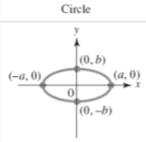
Graph hyperbolas by using asymptotes.

$(x - h)^2 + (y - k)^2 = r^2$	 Circle	The center is (h, k), and the radius is r.	x^2- and y^2-terms have the same positive coefficient.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	 Ellipse	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The y-intercepts are $(0, b)$ and $(0, -b)$.	x^2- and y^2-terms have different positive coefficients.

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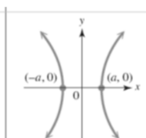
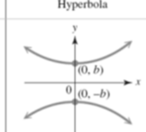
Graph hyperbolas by using asymptotes.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	 Hyperbola	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	x^2 has a positive coefficient. y^2 has a negative coefficient.
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$	 Hyperbola	The y-intercepts are $(0, b)$ and $(0, -b)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	y^2 has a positive coefficient. x^2 has a negative coefficient.

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CLASSROOM EXAMPLE 3 — **Identifying the Graphs of Equations**

Identify the graph of the equation.

$3x^2 = 27 - 4y^2$

Solution:

$$3x^2 + 4y^2 = 27$$

$$\frac{x^2}{9} + \frac{4y^2}{27} = 1$$

Since the x^2 and y^2 terms have different positive coefficients, the graph of the equation is an ellipse.

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CLASSROOM EXAMPLE 3 — **Identifying the Graphs of Equations (cont'd)**

Identify the graph of each equation.

Solution:

$6x^2 = 100 + 2y^2$

$$6x^2 - 2y^2 = 100$$

$$\frac{x^2}{\frac{50}{3}} - \frac{y^2}{50} = 1$$

Because of the minus sign and since both variables are squared, the graph of the equation is a hyperbola.

$3x^2 = 27 - 4y$

$$4y = -3x^2 + 27$$

This is an equation of a vertical parabola since only one variable, x, is squared.

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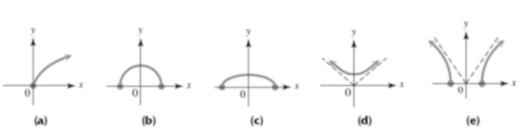
Graph certain square root functions.

Recall that no vertical line will intersect the graph of a function in more than one point.

Horizontal parabolas, all circles, ellipses, and most hyperbolas do not satisfy the conditions of a function. However, by considering only a part of the graph of each of these we have the graph of a function.



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Graph certain square root functions.

Square Root Function

For an algebraic expression u, with $u \geq 0$, a function of the form

$$f(x) = \sqrt{u}$$

is called a **generalized square root function.**

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CLASSROOM EXAMPLE 4 — **Graphing a Semicircle**

Graph $f(x) = \sqrt{4 - x^2}$. Give the domain and range.

Solution:

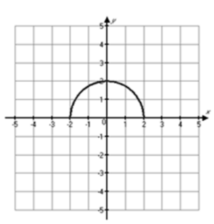
Replace $f(x)$ with y and square both sides.

$$y = \sqrt{4 - x^2}$$

$$y^2 = 4 - x^2$$

$$x^2 + y^2 = 4$$

The equation is the graph of a circle with center (0, 0) and radius 2. We only want the principal square root of the original equation and its graph is the upper half of the circle.



Domain: [–2, 2]

Range: [0, 2]

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CLASSROOM EXAMPLE 5 — **Graphing a Portion of an Ellipse**

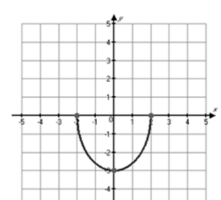
Graph $\frac{y}{3} = -\sqrt{1 - \frac{x^2}{4}}$. Give the domain and range.

Solution:

$$\frac{y^2}{9} = 1 - \frac{x^2}{4}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

The equation is of an ellipse with intercepts (2, 0), (–2, 0), (0, 3), and (0, –3). We need the nonpositive, or lower half of the ellipse.



Domain: [–2, 2]

Range: [–3, 0]

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