12.4 The Binomial Theorem

Objectives

1 Expand a binomial raised to a power.

2 Find any specified term of the expansion of a binomial.

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Expand a binomial raised to a power.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
and so on

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Expand a binomial raised to a power.

***n* factorial (*n*!)**

For any positive integer *n*,

$$n! = n(n-1)(n-2)(n-3)\cdots(2)(1).$$

By definition, $0! = 1$

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CLASSROOM EXAMPLE 1 **Evaluating Factorials**

Evaluate 4!.

Solution:

$= 4 \cdot 3 \cdot 2 \cdot 1$

$= 24$

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CLASSROOM EXAMPLE 2 **Evaluating Expressions Involving Factorials**

Find the value of each expression.

Solution:

$$\frac{7!}{6!1!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(6\cdot5\cdot4\cdot3\cdot2\cdot1)(1)} = \frac{7}{1} = 7$$

$$\frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(4\cdot3\cdot2\cdot1)} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = \frac{7\cdot5}{1} = 35$$

$$\frac{7!}{2!5!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(5\cdot4\cdot3\cdot2\cdot1)} = \frac{7\cdot6}{2\cdot1} = \frac{7\cdot3}{1} = 21$$

$$\frac{7!}{7!0!} = \frac{1}{0!} = \frac{1}{1} = 1$$

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Expand a binomial raised to a power.

Formula for the Binomial Coefficient $_nC_r$

For nonnegative integers *n* and *r*, where $r \le n$,

$$_nC_r = \frac{n!}{r!(n-r)!}.$$

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CLASSROOM EXAMPLE 3 — **Evaluating Binomial Coefficients**

Evaluate ${}_8C_5$.

Solution:

$$= \frac{8!}{5!(8-5)!}$$

$$= \frac{8!}{5!3!}$$

$$= \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1!}{(5\cdot 4\cdot 3\cdot 2\cdot 1)(3\cdot 2\cdot 1)}$$

$$= \frac{40320}{720}$$

$$= 56$$

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Expand a binomial raised to a power.

Binomial Theorem

For any positive integer n,

$$(x+y)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}y + \frac{n!}{(n-2)!2!}x^{n-2}y^2 + \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \cdots + \frac{n!}{(n-1)!1!}xy^{n-1} + y^n.$$

The binomial theorem can be written in summation notation as

$$(x+y)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!}x^{n-i}y^i.$$

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CLASSROOM EXAMPLE 4 — **Using the Binomial Theorem**

Expand $(x^2 + 3)^5$.

Solution:

$$= (x^2)^5 + \frac{5!}{4!1!}(x^2)^4(3)^1 + \frac{5!}{3!2!}(x^2)^3(3)^2 + \frac{5!}{2!3!}(x^2)^2(3)^3 + \frac{5!}{1!4!}(x^2)^1(3)^4 + 3^5$$

$$= x^{10} + 5x^8(3)^1 + 10x^6(9) + 10x^4(27) + 5x^2(81) + 243$$

$$= x^{10} + 15x^8 + 90x^6 + 270x^4 + 405x^2 + 243$$

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CLASSROOM EXAMPLE 5 — **Using the Binomial Theorem**

Expand $\left(\frac{x}{4} - 3y\right)^4$.

Solution:

$$= \left(\frac{x}{4}\right)^4 + \frac{4!}{3!1!}\left(\frac{x}{4}\right)^3(-3y)^1 + \frac{4!}{2!2!}\left(\frac{x}{4}\right)^2(-3y)^2 + \frac{4!}{1!3!}\left(\frac{x}{4}\right)^1(-3y)^3 + (-3y)^4$$

$$= \frac{x^4}{4^4} + 4\cdot\frac{x^3}{4^3}(-3y) + 6\cdot\frac{x^2}{4^2}(9y^2) + 4\cdot\frac{x}{4}(-27y^3) + 81y^4$$

$$= \frac{1}{256}x^4 - \frac{3}{16}x^3y + \frac{27}{8}x^2y^2 - 27xy^3 + 81y^4$$

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Find any specified term of the expansion of a binomial.

***r*th Term of the Binomial Expansion**

If $n \geq r - 1$, then the rth term of the expansion of $(x + y)^n$ is

$$\frac{n!}{(r-1)![n-(r-1)]!}x^{n-(r-1)}y^{r-1}.$$

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CLASSROOM EXAMPLE 6 — **Finding a Single Term of a Binomial Expansion**

Find the fifth term of $\left(\frac{x}{2} - y\right)^9$.

Solution:

In the fifth term of $\left(\frac{x}{2} - y\right)^9$, the exponent on $(-y)$ is 5 – 1 = 4 and the exponent on $\left(\frac{x}{2}\right)$ is 9 – 4 = 5.

The fifth term is

$$= \frac{9!}{5!4!}\left(\frac{x}{2}\right)^5(-y)^4 = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{x^5}{2^5}\cdot y^4$$

$$= 126\cdot\frac{1}{32}x^5y^4$$

$$= \frac{63}{16}x^5y^4$$

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