12.1 Sequences and Series

Objectives

1. Find the terms of a sequence, given the general term.
2. Find the general term of a sequence.
3. Use sequences to solve applied problems.
4. Use summation notation to evaluate a series.
5. Write a series with summation notation.
6. Find the arithmetic mean (average) of a group of numbers.

Sequences and Series

A **sequence** is a function whose domain is the set of a natural numbers.

Infinite Sequence

An **infinite sequence** is a function with the set of all positive integers as the domain. A **finite sequence** is a function with domain of the form {1, 2, 3, …, n}.

Objective 1

Find the terms of a sequence, given the general term.

CLASSROOM EXAMPLE 1 **Writing the Terms of Sequences from the General Term**

Given an infinite sequence with $a_n = 2n + 1$, find a_4.

Solution:

$a_n = 2n + 1$

To get a_4, the fourth term, replace n with 4.

$a_4 = 2(4) + 1 = 9$

Objective 2

Find the general term of a sequence.

CLASSROOM EXAMPLE 2 **Finding the General Term of a Sequence**

Find an expression for the general term a_n of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Solution:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$ **Can be written as**

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots, \text{ so } a_n = \frac{1}{2^n}, \text{ or } a_n = \left(\frac{1}{2}\right)^n.$$

Objective 3

Use sequences to solve applied problems.

CLASSROOM EXAMPLE 3 **Using a Sequence in an Application**

Saad Alarachi borrows $10,000 and agrees to pay a monthly payment of $1000 plus 2% of the unpaid balance.

Solution:

Month	Interest	Payment	Unpaid balance
0			10,000
1	10,000(0.02) = 200	1000 + 200 = 1200	10,000 – 1000 = 9000
2	9000(0.02) = 180	1000 + 180 = 1180	9000 – 1000 = 8000
3	8000(0.02) = 160	1000 + 160 = 1160	8000 – 1000 = 7000
4	7000(0.02) = 140	1000 + 140 = 1140	7000 – 1000 = 6000

The payments are $1200, $1180, $1160, and $1140; the unpaid balance is $6000.

Objective 4

Use summation notation to evaluate a series.

Use summation notation to evaluate a series.

Series

The indicated sum of the terms of a sequence is called a **series.**

We use a compact notation, called **summation notation,** to write a series from the general term of the corresponding sequence. The Greek letter sigma (Σ) is used to denote summation.

For example, the sum of the first six terms of the sequence with general term $a_n = 3_n + 2$ is written as

$$\sum_{i=1}^{6}(3i+2).$$

CAUTION The use of *i* as the index of summation has no connection with the complex number *i*.

CLASSROOM EXAMPLE 4 **Evaluating Series Written in Summation Notation**

Write out the terms and evaluate the series.

$$\sum_{i=1}^{5}(2i-4)$$

Solution:

$= (2\cdot1 - 4) + (2\cdot2 - 4) + (2\cdot3 - 4) + (2\cdot4 - 4) + (2\cdot5 - 4)$

$= (2 - 4) + (4 - 4) + (6 - 4) + (8 - 4) + (10 - 4)$

$= -2 + 0 + 2 + 4 + 6$

$= 10$

CLASSROOM EXAMPLE 4 **Evaluating Series Written in Summation Notation (cont'd)**

Write out the terms and evaluate the series.

$$\sum_{i=2}^{6}(i+1)(i-2)$$

Solution:

$= (2 + 1)(2 - 2) + (3 + 1)(3 - 2) + (4 + 1)(4 - 2) +$
$(5 + 1)(5 - 2) + (6 + 1)(6 - 2)$

$= 3(0) + 4(1) + 5(2) + 6(3) + 7(4)$

$= 0 + 4 + 10 + 18 + 28$

$= 60$

CLASSROOM EXAMPLE 4 — **Evaluating Series Written in Summation Notation (cont'd)**

Write out the terms and evaluate each series.

$$\sum_{i=1}^{4}(2i)^2$$

Solution:

$= (2 \cdot 1)^2 + (2 \cdot 2)^2 + (2 \cdot 3)^2 + (2 \cdot 4)^2$

$= 2^2 + 4^2 + 6^2 + 8^2$

$= 4 + 16 + 36 + 64$

$= 120$

Objective 5

Write a series with summation notation.

CLASSROOM EXAMPLE 5 — **Writing Series with Summation Notation**

Write each with summation notation.

3 + 12 + 27 + 48 + 75

Solution:

$= 3(1 + 4 + 9 + 16 + 25)$

$= 3(1^2 + 2^2 + 3^2 + 4^2 + 5^2)$

$$= \sum_{i=1}^{5} 3i^2$$

7 + 12 + 17 + 22

The terms increase by 5, so $5i$ is part of the general term. If $i = 1$, $5i = 5$, but since the first term is 7, we must add 2 to $5i$ to get 7. There are four terms, so

$$= \sum_{i=1}^{4}(5i+2)$$

Objective 6

Find the arithmetic mean (average) of a group of numbers.

Find the arithmetic mean (average) of a group of numbers.

Arithmetic Mean or Average

The **arithmetic mean**, or **average**, of a group of numbers is symbolized $\bar{x}$ and is found by dividing the sum of the numbers by the number of numbers. That is,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

The values of x_i represent the individual numbers in the group, and n represents the number of numbers.

CLASSROOM EXAMPLE 6 — **Finding the Arithmetic Mean, or Average**

The following table shows the number of FDIC-insured financial institutions for each year during the period from 2002 through 2008. What is the average number of institutions per year for the 5-yr period 2004-2008?

Year	Number of Institutions
2002	9369
2003	9194
2004	8988
2005	8845
2006	8691
2007	8544
2008	8314

Source: U.S. Federal Deposit Insurance Corporation.

Solution:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{5} x_i}{5}$$

$$= \frac{8988 + 8845 + 8691 + 8544 + 8314}{5}$$

$$= \frac{43382}{5} = 8676.4 \quad \text{or} \quad \approx 8676 \text{ institutions}$$

12.2 Arithmetic Sequences

Objectives

1 Find the common difference of an arithmetic sequence.

2 Find the general term of an arithmetic sequence.

3 Use an arithmetic sequence in an application.

4 Find any specified term or the number of terms of an arithmetic sequence.

5 Find the sum of a specified number of terms of an arithmetic sequence.

PEARSON

Find the common difference of an arithmetic sequence.

Arithmetic Sequence

An **arithmetic sequence**, or **arithmetic progression,** is a sequence in which each term after the first is found by adding a constant number to the preceding term.

CLASSROOM EXAMPLE 1 **Finding the Common Difference**

Find *d* for the arithmetic sequence

$$1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$$

Solution:

You should find the difference for all pairs of adjacent terms to determine if the sequence is arithmetic. In this case, we are given that the sequence is arithmetic, so *d* is the difference between any two adjacent terms. Choose the terms $\frac{5}{3}$ and $\frac{4}{3}$.

$$d = \frac{5}{3} - \frac{4}{3} = \frac{1}{3}$$

CLASSROOM EXAMPLE 2 **Writing the Terms of a Sequence from the First Term and the Common Difference**

Write the first five terms of the arithmetic sequence with first term 5 and common difference ½ .

Solution:

Given $a_1 = 5$ and $d = ½$,

$$a_2 = a_1 + d = 5 + ½ = 5 ½$$
$$a_3 = a_2 + d = 5 ½ + ½ = 6$$
$$a_4 = a_3 + d = 6 + ½ = 6 ½$$
$$a_5 = a_4 + d = 6 ½ + ½ = 7$$

The first five terms of the sequence are 5, 5 ½ , 6, 6 ½ , 7.

Find the general term of an arithmetic sequence.

General Term of an Arithmetic Sequence

The general term of an arithmetic sequence with first term a_1 and common difference *d* is

$$a_n = a_1 + (n - 1)d.$$

CLASSROOM EXAMPLE 3 **Finding the General Term of an Arithmetic Sequence**

Find the general term of the arithmetic sequence 4, 2, 0, –2, …

Solution:

To find *d*, subtract any two adjacent terms.

$$d = -2 - 0 = -2$$

The first term is $a_1 = 4$.

Now find a_n.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 4 + (n-1)(-2) \\ &= 4 - 2n + 2 \\ &= -2n + 6 \end{aligned}$$

Thus $a_{20} = -2(20) + 6 = -40 + 6 = -34$.

CLASSROOM EXAMPLE 4 **Applying an Arithmetic Sequence**

How much will be in an account if an initial deposit of $5000 is followed by a $250 contribution each month for 36 months?

Solution:

After 1 month, the account will have

$5000 + 1 • $250 = $5250.

After 2 months, the account will have

$5000 + 2 • $250 = $5500.

In general, after *n* months the account will have

$5000 + *n* • $250.

Thus, after 36 months, the account will have

$5000 + 36 • $250 = $14,000.

Objective 4

Find any specified term or the number of terms of an arithmetic sequence.

CLASSROOM EXAMPLE 5 **Finding Specified Terms in Sequence**

Find the indicated term for the arithmetic sequence.

Given $a_1 = -15$ and $d = -4$, find a_{12}.

Solution:

$$a_n = a_1 + (n-1)d$$

$$a_{12} = a_1 + (12-1)d$$

$$= -15 + 11(-4)$$

$$= -59$$

CLASSROOM EXAMPLE 5 **Finding Specified Terms in Sequence (cont'd)**

Find the indicated term for the arithmetic sequence.

Given $a_3 = 2$ and $a_{10} = 23$, find a_{15}.

Solution:

Use $a_n = a_1 + (n-1)d$ to write a system of equations.

$$a_3 = a_1 + (3-1)d$$

$$2 = a_1 + 2d$$

$$a_{10} = a_1 + (10-1)d$$

$$23 = a_1 + 9d$$

To eliminate a_1, multiply (1) by –1 and add the result to (2).

$$\begin{aligned} -2 &= -a_1 - 2d \\ 23 &= a_1 + 9d \\ \hline 21 &= 7d \\ 3 &= d \end{aligned}$$

CLASSROOM EXAMPLE 5 **Finding Specified Terms in Sequence (cont'd)**

From (1), $2 = a_1 + 2(3)$, so $a_1 = -4$. Now find a_{15}.

$$a_{15} = a_1 + (15-1)d$$

$$= -4 + 14(3)$$

$$= 38$$

There are 5 differences from a_{10} to a_{15}, so

$$a_{15} = a_{10} + 5d$$

$$= 23 + 5(3)$$

$$= 38$$

CLASSROOM EXAMPLE 6 **Finding the Number of Terms in a Sequence**

Find the number of terms in the arithmetic sequence 8, 5, 2, –1, …, –46.

Solution:

$a_n = a_1 + (n-1)d$	**Formula for a_n**
$-46 = 8 + (n-1)(-3)$	**$d = 5 - 8 = -3$**
$-46 = 8 - 3n + 3$	**Distributive property**
$-57 = -3n$	**Simplify**
$19 = n$	**Divide by –3.**

The sequence has 19 terms.

Objective 5

Find the sum of a specified number of terms of an arithmetic sequence.

CLASSROOM EXAMPLE 7 — **Finding the Sum of the First *n* Terms of an Arithmetic Sequence**

Find the sum of the first nine terms of the arithmetic sequence in which $a_n = 5 + 2n$.

Solution:

Since we want the sum of the first nine terms, we'll find a_1 and a_9 using $a_n = 5 + 2n$.

$$a_1 = 5 + 2(1) = 7$$

$$a_9 = 5 + 2(9) = 23$$

CLASSROOM EXAMPLE 7 — **Finding the Sum of the First *n* Terms of an Arithmetic Sequence (cont'd)**

Now use the formula for the sum of the first *n* terms of an arithmetic sequence.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{9}{2}(a_1 + a_9)$$

$$= \frac{9}{2}(7 + 23)$$

$$= \frac{9}{2}(30)$$

$$= 135$$

Find the sum of specified number of terms of an arithmetic sequence.

Sum of the First *n* Terms of an Arithmetic Sequence

The sum of the first *n* terms of the arithmetic sequence with the first term a_1, nth term a_n, and common difference d is given by either formula

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{or} \quad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right].$$

CLASSROOM EXAMPLE 8 — **Finding the Sum of the First n Terms of an Arithmetic Sequence**

Find the sum of the first 10 terms of the arithmetic sequence having first term –7 and common difference 3.

Solution:

We are given $a_1 = -7$, $d = 3$, and $n = 10$. Use the second formula for the sum of the arithmetic sequence.

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$S_{10} = \frac{10}{2}\left[2(-7) + (10-1)3\right]$$

$$= 5\left[-14 + (9)3\right]$$

$$= 5(-14 + 27)$$

$$= 65$$

CLASSROOM EXAMPLE 9 — **Using S_n to Evaluate a Summation**

Evaluate $\sum_{i=1}^{20}(4i+1)$.

Solution:

To find the first and last (20th) terms, let $n = 1$ and $n = 20$ and $a_n = 4n + 1$.

$$a_1 = 4(1) + 1 = 5$$

$$a_{20} = 4(20) + 1 = 81$$

Now find S_{20}.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{20} = \frac{20}{2}(5 + 81)$$

$$= 10(86)$$

$$= 860$$

12.3 Geometric Sequences

Objectives

1 Find the common ratio of a geometric sequence.

2 Find the general term of a geometric sequence.

3 Find any specified term of a geometric sequence.

4 Find the sum of a specified number of terms of a geometric sequence.

5 Apply the formula for the future value of an ordinary annuity.

6 Find the sum of an infinite number of terms of a certain geometric sequences.

PEARSON

Geometric Sequences

Geometric Sequence

A **geometric sequence**, or **geometric progression**, is a sequence in which each term after the first is found by multiplying the preceding term by a nonzero constant.

Find the common ratio of a geometric sequence.

We find the constant multiplier, called the **common ratio,** by dividing any term after the first by the preceding term. That is, the common ratio is

$$r = \frac{a_{n+1}}{a_n}.$$

CLASSROOM EXAMPLE 1 **Finding the Common Ratio**

Determine r for the geometric sequence $1, -2, 4, -8, 16, \ldots$

Solution:

You should find the ratio for all pairs of adjacent terms to determine if the sequence is geometric. In this case, we are ***given*** that the sequence is geometric, so to find r, choose any two adjacent terms and divide the second one by the first one. Choose the terms 1 and –2.

$$r = \frac{-2}{1} = -2$$

Notice that any other two adjacent terms could have been used with the same result.

The common ration is $r = -2$.

Find the general term of a geometric sequence.

General Term of a Geometric Sequence

The general term of the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

CAUTION In finding $a_1 r^{n-1}$, be careful to use the correct order of operations. The value of r^{n-1} must be found first. Then multiply the result by a_1.

CLASSROOM EXAMPLE 2 **Finding the General Term of a Geometric Sequence**

Find the general term of the sequence –2, –8, –32, ….

Solution:

The common ratio is $r = \frac{-8}{-2} = 4.$

Substitute into the formula for a_n.

$$a_n = a_1 r^{n-1}$$
$$= -2(4)^{n-1}$$

CLASSROOM EXAMPLE 3 — **Finding Specified Terms in Sequences**

Find the indicated term for each geometric sequence.

Solution:

$a_1 = -2$, $r = 5$; a_6

$a_n = a_1 r^{n-1}$

$a_6 = -2(5)^{6-1}$

$= -2(5)^5$

$= -6250$

4, 28, 196, 1372, …; a_8

The common ratio is $r = \frac{28}{4} = 7.$

Substitute into the formula for a_n.

$a_n = a_1 r^{n-1}$

$a_8 = 4(7)^{8-1}$

$= 3{,}294{,}172$

CLASSROOM EXAMPLE 4 — **Writing the Terms of a Sequence**

Write the first five terms of the geometric sequence whose first term is 3 and whose common ratio is –2.

Solution:

$a_1 = 3$

$a_2 = a_1 r = 3(-2) = -6$

$a_3 = a_2 r = -6(-2) = 12$

$a_4 = a_3 r = 12(-2) = -24$

$a_5 = a_4 r = -24(-2) = 48$

Objective 4

Find the sum of a specified number of terms of a geometric sequence.

Find the sum of a specified number of terms of a geometric sequence.

Sum of the First *n* Terms of a Geometric Sequence

The sum of the first *n* terms of the geometric sequence with the first term a_1 and common ratio *r* is

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \quad (r \neq 1).$$

CLASSROOM EXAMPLE 5 — **Finding the Sum of the First *n* Terms of a Geometric Sequence**

Evaluate the sum of the first six terms of the geometric sequence with first term 6 and common ratio 3.

Solution:

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$S_6 = \frac{6(3^6 - 1)}{3 - 1}$$

$$= \frac{6(729 - 1)}{2}$$

$$= 3(728)$$

$$= 2184$$

CLASSROOM EXAMPLE 6 — **Using the Formula for S_n to Find a Summation**

Evaluate $\sum_{i=1}^{6} 3\left(\frac{1}{4}\right)^i$.

Solution:

For $\sum_{i=1}^{6} 3\left(\frac{1}{4}\right)^i$, $a_1 = \frac{3}{4}$ and $r = \frac{1}{4}$. Find S_6.

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$S_6 = \frac{\frac{3}{4}\left[\left(\frac{1}{4}\right)^6 - 1\right]}{\frac{1}{4} - 1}$$

$$= \frac{\frac{3}{4}\left[\frac{1}{4096} - 1\right]}{-\frac{3}{4}}$$

$$= -1\left(\frac{1}{4096} - \frac{4096}{4096}\right)$$

$$= -1\left(\frac{-4095}{4096}\right)$$

$$= \frac{4095}{4096} \text{ or } \approx 0.9998$$

Apply the formula for the future value of an ordinary annuity.

Future Value of an Ordinary Annuity

The future value for an ordinary annuity is

$$S = R\left[\frac{(1+i)^n - 1}{i}\right],$$

where S is the future value,
R is the payment at the end of each period,
i is the interest rate per period, and
n is the number of periods.

CLASSROOM EXAMPLE 7 **Applying the Formula for the Future Value of an Annuity**

Sonny's Building Specialties deposits $2500 at the end of each year into an account paying 4% per yr, compounded annually. Find the total amount on deposit after 10 yr.

Solution:

$$S = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$S = 2500\left[\frac{(1+0.04)^{10} - 1}{0.04}\right]$$

$$= 30{,}015.268$$

The future value of the annuity is $30,015.27.

CLASSROOM EXAMPLE 7 **Applying the Formula for the Future Value of an Annuity (cont'd)**

Igor Kalugin is an athlete who believes that his playing career will last 10 yr. How much will be in Igor's account after 10 yr if he deposits $2000 at the end of each month at 6% interest compounded monthly?

Solution:

Now use r = $2000, $i = \frac{0.06}{12} = 0.005,$ and n = 12(10) = 120.

$$S = \$2000\left[\frac{(1+0.005)^{120} - 1}{0.005}\right]$$

$$= \$327{,}758.69$$

Find the sum of an infinite number of terms of certain geometric sequences.

Sum of the Terms of an Infinite Geometric Sequence

The sum S of the terms of an infinite geometric sequence with the first term a_1 and common ratio r, $|r| < 1$, is

$$S = \frac{a_1}{1-r}.$$

If $|r| \geq 1$, then the sum does not exist.

CLASSROOM EXAMPLE 8 **Finding the Sum of the Terms of an Infinite Geometric Sequence**

Find the sum of the terms of the infinite geometric sequence with $a_1 = -2$ and $r = -\frac{5}{8}$.

Solution:

Use the formula for the sum of the terms of an infinite geometric sequence with $a_1 = -2$ and $r = -\frac{5}{8}$.

$$S = \frac{a_1}{1-r} \quad = \frac{-2}{1-\left(-\frac{5}{8}\right)} \quad = \frac{-2}{\frac{13}{8}} \quad = -\frac{16}{13}$$

CLASSROOM EXAMPLE 9 **Finding the Sum of the Terms of an Infinite Geometric Series**

Find $\sum_{i=1}^{\infty}\left(\frac{1}{5}\right)\left(\frac{5}{7}\right)^i$.

Solution:

For $\sum_{i=1}^{\infty}\left(\frac{1}{5}\right)\left(\frac{5}{7}\right)^i, a_1 = \frac{1}{7}$ and $r = \frac{5}{7}$.

$$S = \frac{a_1}{1-r} \quad = \frac{\frac{1}{7}}{1-\frac{5}{7}} \quad = \frac{\frac{1}{7}}{\frac{2}{7}} \quad = \frac{1}{2}$$

12.4 The Binomial Theorem

Objectives

1 Expand a binomial raised to a power.

2 Find any specified term of the expansion of a binomial.

PEARSON

Expand a binomial raised to a power.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
and so on

Expand a binomial raised to a power.

***n* factorial (*n*!)**

For any positive integer *n*,

$$n! = n(n-1)(n-2)(n-3)\cdots(2)(1).$$

By definition, $0! = 1$

CLASSROOM EXAMPLE 1 **Evaluating Factorials**

Evaluate 4!.

Solution:

$= 4 \cdot 3 \cdot 2 \cdot 1$

$= 24$

CLASSROOM EXAMPLE 2 **Evaluating Expressions Involving Factorials**

Find the value of each expression.

Solution:

$$\frac{7!}{6!1!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(6\cdot5\cdot4\cdot3\cdot2\cdot1)(1)} = \frac{7}{1} = 7$$

$$\frac{7!}{3!4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(4\cdot3\cdot2\cdot1)} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = \frac{7\cdot5}{1} = 35$$

$$\frac{7!}{2!5!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(5\cdot4\cdot3\cdot2\cdot1)} = \frac{7\cdot6}{2\cdot1} = \frac{7\cdot3}{1} = 21$$

$$\frac{7!}{7!0!} = \frac{1}{0!} = \frac{1}{1} = 1$$

Expand a binomial raised to a power.

Formula for the Binomial Coefficient $_nC_r$

For nonnegative integers *n* and *r*, where $r \le n$,

$$_nC_r = \frac{n!}{r!(n-r)1!}.$$

CLASSROOM EXAMPLE 3 Evaluating Binomial Coefficients

Evaluate ${}_8C_5$.

Solution:

$$= \frac{8!}{5!(8-5)!}$$

$$= \frac{8!}{5!3!}$$

$$= \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1!}{(5\cdot4\cdot3\cdot2\cdot1)(3\cdot2\cdot1)}$$

$$= \frac{40320}{720}$$

$$= 56$$

Expand a binomial raised to a power.

Binomial Theorem

For any positive integer *n*,

$$(x+y)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}y + \frac{n!}{(n-2)!2!}x^{n-2}y^2 + \frac{n!}{3!(n-3)!}x^{n-3}y^3 + \cdots + \frac{n!}{(n-1)!1!}xy^{n-1} + y^n.$$

The binomial theorem can be written in summation notation as

$$(x+y)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!}x^{n-i}y^i.$$

CLASSROOM EXAMPLE 4 Using the Binomial Theorem

Expand $(x^2 + 3)^5$.

Solution:

$$= (x^2)^5 + \frac{5!}{4!1!}(x^2)^4(3)^1 + \frac{5!}{3!2!}(x^2)^3(3)^2 + \frac{5!}{2!3!}(x^2)^2(3)^3 + \frac{5!}{1!4!}(x^2)^1(3)^4 + 3^5$$

$$= x^{10} + 5x^8(3)^1 + 10x^6(9) + 10x^4(27) + 5x^2(81) + 243$$

$$= x^{10} + 15x^8 + 90x^6 + 270x^4 + 405x^2 + 243$$

CLASSROOM EXAMPLE 5 Using the Binomial Theorem

Expand $\left(\frac{x}{4} - 3y\right)^4$.

Solution:

$$= \left(\frac{x}{4}\right)^4 + \frac{4!}{3!1!}\left(\frac{x}{4}\right)^3(-3y)^1 + \frac{4!}{2!2!}\left(\frac{x}{4}\right)^2(-3y)^2 + \frac{4!}{1!3!}\left(\frac{x}{4}\right)^1(-3y)^3 + (-3y)^4$$

$$= \frac{x^4}{4^4} + 4\cdot\frac{x^3}{4^3}(-3y) + 6\cdot\frac{x^2}{4^2}(9y^2) + 4\cdot\frac{x}{4}(-27y^3) + 81y^4$$

$$= \frac{1}{256}x^4 - \frac{3}{16}x^3y + \frac{27}{8}x^2y^2 - 27xy^3 + 81y^4$$

Find any specified term of the expansion of a binomial.

***r*th Term of the Binomial Expansion**

If $n \geq r - 1$, then the *r*th term of the expansion of $(x + y)^n$ is

$$\frac{n!}{(r-1)![n-(r-1)]!}x^{n-(r-1)}y^{r-1}.$$

CLASSROOM EXAMPLE 6 Finding a Single Term of a Binomial Expansion

Find the fifth term of $\left(\frac{x}{2} - y\right)^9$.

Solution:

In the fifth term of $\left(\frac{x}{2} - y\right)^9$, the exponent on $(-y)$ is 5 – 1 = 4 and the exponent on $\left(\frac{x}{2}\right)$ is 9 – 4 = 5.

The fifth term is $$= \frac{9!}{5!4!}\left(\frac{x}{2}\right)^5(-y)^4 = \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}\cdot\frac{x^5}{2^5}\cdot y^4$$

$$= 126\cdot\frac{1}{32}x^5y^4$$

$$= \frac{63}{16}x^5y^4$$