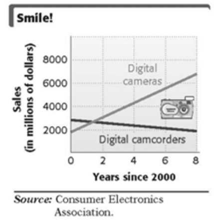
4.1 Systems of Linear Equations in Two Variables

Objectives

1. Decide whether an ordered pair is a solution of a linear system.
2. Solve linear systems by graphing.
3. Solve linear systems (with two equations and two variables) by substitution.
4. Solve linear systems (with two equations and two variables) by elimination.
5. Solve special systems.

PEARSON

Systems of Linear Equations in Two Variables



We can use linear equations to model the graphs of real applications. In this case, the sales of digital cameras and camcorders.

The two straight-line graphs intersect at the point in time when the two products had the ***same*** sales.

(Here, x = 0 represents 2000, x = 1 represents 2001, and so on; y represents sales in millions of dollars.)

A set of equations is called a **system of equations,** in this case, a **linear system of equations.**

Decide whether an ordered pair is a solution of a linear system.

The **solution set of a linear system** of equations contains all ordered pairs that satisfy all the equations of the system ***at the same time.***

CLASSROOM EXAMPLE 1 **Deciding Whether an Ordered Pair is a Solution**

Is the ordered pair a solution of the given system?

(–4, 2)

$$2x + y = -6$$
$$x + 3y = 2$$

Solution:

Replace x with –4 and y with 2 in each equation of the system.

$$2x + y = -6 \qquad x + 3y = 2$$
$$2(-4) + 2 = -6 \qquad -4 + 3(2) = 2$$
$$-8 + 2 = -6 \qquad -4 + 6 = 2$$
$$-6 = -6 \qquad 2 = 2$$

True **True**

Since (–4, 2) makes both equations true, it is a solution.

CLASSROOM EXAMPLE 1 **Deciding Whether an Ordered Pair is a Solution (cont’d)**

Is the ordered pair a solution of the given system?

(3, –12)

$$2x + y = -6$$
$$x + 3y = 2$$

Solution:

Replace x with 3 and y with −12 in each equation of the system.

$$2x + y = -6 \qquad x + 3y = 2$$
$$2(3) + (-12) = -6 \qquad 3 + 3(-12) = 2$$
$$6 - 12 = -6 \qquad 3 - 36 = 2$$
$$-6 = -6 \qquad -33 = 2$$

True **False**

The ordered pair (3, –12) is not a solution of the system, since it does not make ***both*** equations true.

CLASSROOM EXAMPLE 2 **Solving a System by Graphing**

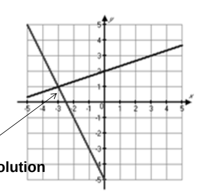
Solve the system of equations by graphing.

$$2x + y = -5$$
$$-x + 3y = 6$$

Solution:

Graph each linear equation.

$$2x + y = -5 \qquad -x + 3y = 6$$
$$y = -2x - 5 \qquad y = \frac{1}{3}x + 2$$



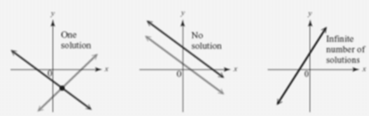
The graph suggests that the point of intersection is the ordered pair (–3, 1).

Check the solution in ***both*** equations.

Solve linear systems by graphing.

Graphs of Linear Systems in Two Variables

1. **The two graphs intersect in a single point.** The coordinates of this point give the only solution of the system. Since the system has a solution, it is **consistent.** The equations are *not* equivalent, so they are **independent.** See Figure 3(a).
2. **The graphs are parallel lines.** There is no solution common to both equations, so the solution set is ∅ and the system is **inconsistent.** Since the equations are *not* equivalent, they are **independent.** See Figure 3(b).
3. **The graphs are the same line.** Since any solution of one equation of the system is a solution of the other, the solution set is an infinite set of ordered pairs representing the points on the line. This type of system is **consistent** because there is a solution. The equations are equivalent, so they are **dependent.** See Figure 3(c).



Consistent system; independent equations (a) | Inconsistent system; independent equations (b) | Consistent system; dependent equations (c)

FIGURE 3

Solve linear systems (with two equations and two variables) by substitution.

It can be difficult to read exact coordinates, especially if they are not integers, from a graph. For this reason, we usually use algebraic methods to solve systems.

The **substitution method,** is most useful for solving linear systems in which one equation is solved or can easily be solved for one variable in terms of the other.

CLASSROOM EXAMPLE 3 **Solving a System by Substitution**

Solve the system. $5x - 3y = -6 \quad (1)$

Solution: $x = 2 - y \quad (2)$

Since equation (2) is solved for *x*, substitute 2 – *y* for *x* in equation (1).

$$5x - 3y = -6 \qquad (1)$$
$$5(2 - y) - 3y = -6$$
$$10 - 5y - 3y = -6 \qquad \textbf{Distributive property}$$
$$10 - 8y = -6 \qquad \textbf{Combine like terms.}$$
$$-8y = -16 \qquad \textbf{Subtract 10.}$$
$$y = 2 \qquad \textbf{Divide by – 8.}$$

Be sure to use parentheses here.

CLASSROOM EXAMPLE 3 **Solving a System by Substitution (cont'd)**

$$5x - 3y = -6 \qquad (1)$$
$$x = 2 - y \qquad (2)$$

We found *y*. Now find *x* by substituting 2 for *y* in equation (2).

$$x = 2 - y$$
$$= 2 - 2 = 0$$

Thus *x* = 0 and *y* = 2, giving the ordered pair (0, 2). Check this solution in both equations of the original system.

***Check*:**

$$5x - 3y = -6 \quad (1) \qquad\qquad x = 2 - y \quad (2)$$
$$5(0) - 3(2) = -6 \qquad\qquad 0 = 2 - 2$$
$$0 - 6 = -6 \qquad\qquad 0 = 0$$
$$-6 = -6 \qquad\qquad \textbf{True}$$

True

The solution set is (0, 2).

Solve linear systems (with two equations and two variables) by substitution.

Solving a Linear System by Substitution

Step 1 **Solve one of the equations for either variable.** If one of the equations has a variable term with coefficient 1 or –1, choose it, since the substitution method is usually easier this way.

Step 2 **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

Step 3 **Solve** the equation from ***Step 2.***

Step 4 **Find the other value**. Substitute the result from ***Step 3*** into the equation from ***Step 1*** to find the value of the other variable.

Step 5 **Check** the solution in ***both*** of the ***original*** equations. Then write the solution set.

CLASSROOM EXAMPLE 4 **Solving a System by Substitution**

Solve the system. $4x + y = 5 \quad (1)$

Solution: $2x - 3y = 13 \quad (2)$

Step 1 **Solve one of the equations for either *x* or *y*.**

$$4x + y = 5 \qquad (1)$$
$$y = 5 - 4x \qquad \textbf{Subtract 4}\boldsymbol{x}\textbf{.}$$

Step 2 **Substitute** 5 – 4*x* into equation for *y* (2).

$$2x - 3(5 - 4x) = 13$$

Step 3 **Solve.**

$$2x - 15 + 12x = 13$$
$$14x = 28$$
$$x = 2$$

CLASSROOM EXAMPLE 4 **Solving a System by Substitution (cont'd)**

Step 4 **Now find *y*.**

$$4x + y = 5 \quad (1)$$
$$2x - 3y = 13 \quad (2)$$

$$y = 5 - 4x$$
$$y = 5 - 4(2) = -3$$

Step 5 **Check** the solution (2, –3) in both equations.

$4x + y = 5 \quad (1)$	$2x - 3y = 13 \quad (2)$
$4(2) + (-3) = 5$	$2(2) - 3(-3) = 13$
$8 - 3 = 5$	$4 + 9 = 13$
$5 = 5$	$13 = 13$
True	**True**

The solution set is (2, –3).

CLASSROOM EXAMPLE 5 **Solving a System with Fractional Coefficients**

Solve the system.
$$-2x + 5y = 22 \quad (1)$$
$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{2} \quad (2)$$

Solution:

Clear the fractions in equation (2). Multiply by the LCD, 4.

$$4\left(\frac{1}{2}x + \frac{1}{4}y\right) = 4\left(\frac{1}{2}\right)$$
$$4 \cdot \frac{1}{2}x + 4 \cdot \frac{1}{4}y = 4 \cdot \frac{1}{2}$$
$$2x + y = 2 \quad (3)$$

Solve equation (3) for *y*.
$$2x + y = 2 \quad (3)$$
$$y = 2 - 2x$$

CLASSROOM EXAMPLE 5 **Solving a System with Fractional Coefficients (cont'd)**

$$-2x + 5y = 22 \quad (1)$$
$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{2} \quad (2)$$

Substitute $y = 2 - 2x$ for *y* in equation (1).

$$-2x + 5y = 22$$
$$-2x + 5(2 - 2x) = 22$$
$$-2x + 10 - 10x = 22$$
$$-12x + 10 = 22$$
$$-12x = 12$$
$$x = -1$$

Solve *y*.

$$y = 2 - 2x$$
$$y = 2 - 2(-1)$$
$$y = 2 + 2 = 4$$

A check verifies that the solution set is {(– 1, 4)}.

CLASSROOM EXAMPLE 6 **Solving a System by Elimination**

Solve the system.
$$-2x + 3y = -10 \quad (1)$$
$$2x + 2y = 5 \quad (2)$$

Solution:

Adding the equations together will eliminate *x*.

$$\begin{array}{rl} -2x + 3y = -10 & (1) \\ 2x + 2y = 5 & (2) \\ \hline 5y = -5 & \\ y = -1 & \end{array}$$

To find *x*, substitute –1 for *y* in either equation.

$$2x + 2y = 5 \quad (2)$$
$$2x + 2(-1) = 5$$
$$2x - 2 = 5$$
$$2x = 7 \qquad x = \frac{7}{2}$$

The solution set is $\left\{\left(\frac{7}{2}, -1\right)\right\}$.

Solve linear systems (with two equations and two variables) by elimination.

Solving a Linear System by Elimination

Step 1 **Write both equations in standard form *Ax + By = C*.**

Step 2 **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the *x*- or *y*-terms is 0.

Step 3 **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4 **Solve** the equation from ***Step 3*** for the remaining variable.

Step 5 **Find the other value.** Substitute the result of ***Step 4*** into either of the original equations and solve for the other variable.

Step 6 **Check** the ordered-pair solution in ***both*** of the ***original*** equations. Then write the solution set.

CLASSROOM EXAMPLE 7 **Solving a System by Elimination**

Solve the system.
$$2x + 3y = 19 \quad (1)$$
$$3x - 7y = -6 \quad (2)$$

Solution:

Step 1 **Both equations are in standard form.**

Step 2 **Select a variable to eliminate,** say *y*. Multiply equation (1) by 7 and equation (2) by 3.

Step 3 **Add.**

$$\begin{array}{r} 14x + 21y = 133 \\ 9x - 21y = -18 \\ \hline 23x = 115 \end{array}$$

Step 4 **Solve** for *x*.

$$x = 5$$

CLASSROOM EXAMPLE 7

Solving a System by Elimination (cont'd)

$$2x+3y=19 \quad (1)$$
$$3x-7y=-6 \quad (2)$$

Step 5 **To find *y* substitute** 5 for *x* in either equation (1) or equation (2).

$$2x+3y=19 \quad (1)$$
$$2(5)+3y=19$$
$$10+3y=19$$
$$3y=9$$
$$y=3$$

Step 6 **To check** substitute 5 for *x* and 3 for *y* in both equations (1) and (2).

The ordered pair checks, the solution set is {(5, 3)}.

CLASSROOM EXAMPLE 8

Solving a System of Dependent Equations

Solve the system.

$$2x+y=6 \quad (1)$$
$$-8x-4y=-24 \quad (2)$$

Solution:

Multiply equation (1) by 4 and add the result to equation (2).

$$8x+4y=24 \quad (1)$$
$$-8x-4y=-24 \quad (2)$$
$$0=0 \quad \textbf{True}$$

Equations (1) and (2) are equivalent and have the same graph. The equations are dependent.

The solution set is the set of all points on the line with equation 2*x* + *y* = 6, written in set-builder notation {(*x*, *y*) | 2*x* + *y* = 6}.

CLASSROOM EXAMPLE 9

Solving an Inconsistent System

Solve the system.

$$4x-3y=8 \quad (1)$$
$$8x-6y=14 \quad (2)$$

Solution:

Multiply equation (1) by −2 and add the result to equation (2).

$$-8x+6y=-16 \quad (1)$$
$$8x-6y=14 \quad (2)$$
$$0=-2 \quad \textbf{False}$$

The result of adding the equations is a false statement, which indicates the system is inconsistent. The graphs would be parallel lines. There are no ordered pairs that satisfy both equations.

The solution set is ∅.

Solving special systems.

Special Cases of Linear Systems

If both variables are eliminated when a system of linear equations is solved,

1. there are infinitely many solutions if the resulting statement is ***true*****;**

2. there is no solution if the resulting statement is ***false*****.**

CLASSROOM EXAMPLE 10

Using Slope-Intercept Form to Determine the Number of Solutions

Write each equation in slope-intercept form and then tell how many solutions the system has.

$$3x-6y=9 \quad (1)$$
$$x-2y=3 \quad (2)$$

Solution:

Rewrite both equations in y-intercept form.

$$3x-6y=9 \quad (1)$$
$$-6y=-3x+9$$
$$\frac{-6y}{3}=\frac{-3x+9}{3}$$
$$-2y=-x+3$$
$$y=\frac{1}{2}x-\frac{3}{2}$$

$$x-2y=3 \quad (2)$$
$$-2y=-x+3$$
$$y=\frac{1}{2}x-\frac{3}{2}$$

Both lines have the same slope and same y-intercept. They coincide and therefore have infinitely many solutions.

CLASSROOM EXAMPLE 10

Using Slope-Intercept Form to Determine the Number of Solutions (cont'd)

Write each equation in slope-intercept form and then tell how many solutions the system has.

$$-2x=5y+1 \quad (1)$$
$$-4x=10y+3 \quad (2)$$

Solution:

Rewrite both equations in y-intercept form.

$$-2x=5y+1 \quad (1)$$
$$-2x-1=5y$$
$$-\frac{2}{5}x-\frac{1}{5}=y$$
$$y=-\frac{2}{5}x-\frac{1}{5}$$

$$-4x=10y+3 \quad (2)$$
$$-4x-3=10y$$
$$-\frac{4}{10}x-\frac{3}{10}=y$$
$$y=-\frac{2}{5}x-\frac{3}{10}$$

Both lines have the same slope, but different y-intercepts. They are parallel and therefore have no solutions.