4.2 Systems of Linear Equations in Three Variables

Objectives

1 Understand the geometry of systems of three equations in three variables.

2 Solve linear systems (with three equations and three variables) by elimination.

3 Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.

4 Solve special systems.

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Systems of Linear Equations in Three Variables

A solution of an equation in three variables, such as $2x + 3y - z = 4$ is called an **ordered triple** and is written (x, y, z).

Understand the geometry of system of three equations in three variables.

The graph of a linear equation with three variables is a ***plane,*** not a line.

A number of possible solutions are shown below.

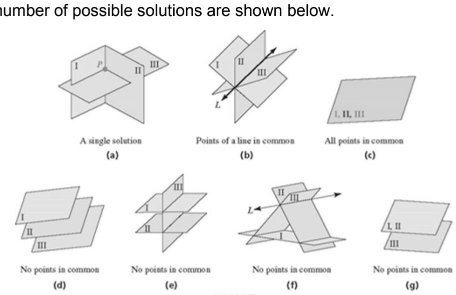

FIGURE 7

Understand the geometry of system of three equations in three variables.

Graphs of Linear Systems in Three Variables

1. **The three planes may meet at a single, common point** that is the solution of the system. (*See* **Figure 7a**).
2. **The three planes may have the points of a line in common,** so that the infinite set of points that satisfy the equation of the line is the solution of the system. (*See* **Figure 7b**).
3. **The three planes may coincide,** so that the solution of the system is the set of all points on a plane. (*See* **Figure 7c**).
4. **The planes may have no points common to all three,** so that there is no solution of the system. (*See* **Figures 7d-g**).

Sole linear systems (with three equations and three variables) by elimination.

In the steps that follow, we use the term **focus variable** to identify the first variable to be eliminated in the process. The focus variable will always be present in the **working equation,** which will be used twice to eliminate this variable.

Solve linear systems (with three equations and three variables) by elimination.

Solving a Linear System in Three Variables

Step 1 **Select a variable and an equation.** A good choice for the variable, which we call the ***focus variable,*** is one that has coefficient 1 or −1. Then select an equation, one that contains the focus variable, as the ***working equation.***

Step 2 **Eliminate the focus variable.** Use the working equation and one of the other two equations of the original system. The result is an equation in two variables.

Step 3 **Eliminate the focus variable again.** Use the working equation and the remaining equation of the original system. The result is another equation in two variables.

Solve linear systems (with three equations and three variables) by elimination.

Solving a Linear System in Three Variables (cont'd)

Step 4 **Write the equations in two variables that result from *Steps 2 and 3* as a system, and solve it.** Doing this gives the values of two of the variables.

Step 5 **Find the value of the remaining variable.** Substitute the values of the two variables found in ***Step 4*** into the working equation to obtain the value of the focus variable.

Step 6 **Check** the ordered-pair solution in ***each*** of the ***original*** equations of the system. Then write the solution set.

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables**

Solve the system.

$$x+y+z=2 \quad (1)$$
$$x-y+2z=2 \quad (2)$$
$$-x+2y-z=1 \quad (3)$$

Solution:

Step 1 Select the variable *y* as the focus variable and equation (1) as the working equation.

$$x+y+z=2 \quad (1)$$

Step 2 Multiplying can be skipped as the focus variable can be eliminated when adding equations (1) and (2)

$$x+y+z=2 \quad (1)$$
$$x-y+2z=2 \quad (2)$$
$$2x \quad +3z=4 \quad (4)$$

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

$$x+y+z=2 \quad (1)$$
$$x-y+2z=2 \quad (2)$$
$$-x+2y-z=1 \quad (3)$$

Step 3 Use the working equation (1), multiply both sides by −2 and add to equation (3) to again eliminate focus variable *y*.

$$-2x-2y-2z=-4 \quad (1)$$
$$-x+2y-z=1 \quad (3)$$
$$-3x \quad -3z=-3 \quad (5)$$

Make sure equation (5) has the same variables as equation 4.

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

$$x+y+z=2 \quad (1)$$
$$x-y+2z=2 \quad (2)$$
$$-x+2y-z=1 \quad (3)$$

Step 4 Write the equations in two variables that result in ***Steps 2 and 3*** as a system, then solve to eliminate *z*. Substitute the value of *x* into equation (4) to sole for *z*.

$$2x+3z=4 \quad (4)$$
$$-3x-3z=-3 \quad (5)$$
$$-x=1 \quad \text{or} \quad x=-1$$

$$2(-1)+3z=4 \quad (4)$$
$$-2+3z=4$$
$$3z=6$$
$$z=2$$

Step 5 Substitute –1 for *x* and 2 for *z* in equation (1) to find *y*.

$$-1+y+2=2 \quad (1)$$
$$y+1=2$$
$$y=1$$

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

Step 6 Check the ordered triple (–1, 1, 2) to be sure the solution satisfies all three equations in the original system.

$$x+y+z=2 \quad (1) \qquad -1+1+2=2$$
$$x-y+2z=2 \quad (2) \qquad -1-1+2(2)=2$$
$$-x+2y-z=1 \quad (3) \qquad -(-1)+2(1)-(2)=1$$

The solution set is {(–1, 1, 2)}.

Write the values of *x, y*, and *z* in the correct order.

CLASSROOM EXAMPLE 2 **Solving a System of Equations with Missing Terms**

Solve the system.

Solution:

$$x-y=6 \quad (1)$$
$$2y+5z=1 \quad (2)$$
$$3x-4z=8 \quad (3)$$

Since equation (3) is missing *y*, eliminate *y* again from equations (1) and (2). Multiply equation (1) by 2 and add the result to equation (2).

$$x-y=6 \quad (1) \qquad 2x-2y \quad =12 \quad (1)\times 2$$
$$2y+5z=1 \quad (2) \qquad 2y+5z \quad =1 \quad (2)$$
$$2x \quad +5z=13 \quad (4)$$

CLASSROOM EXAMPLE 2 — **Solving a System of Equations with Missing Terms (cont'd)**

Use equation (4) together with equation (3) to eliminate *x*. Multiply equation (4) by 3 and equation (3) by –2. Then add the results.

$$\begin{aligned} 2x \quad + 5z &= 13 \quad (4)\times(3) \\ 3x - \quad 4z &= 8 \quad (3)\times(-2) \end{aligned}$$

$$\begin{aligned} 6x + 15z &= 39 \\ -6x + 8z &= -16 \\ \hline 23z &= 23 \\ z &= 1 \end{aligned}$$

CLASSROOM EXAMPLE 2 — **Solving a System of Equations with Missing Terms (cont'd)**

Substitute 1 for *z* in equation (2) to find *y*.

$$\begin{aligned} 2y + 5z &= 1 \quad (2) \\ 2y + 5(1) &= 1 \\ 2y + 5 &= 1 \\ 2y &= -4 \\ y &= -2 \end{aligned}$$

Substitute –2 for *y* in (1) to find *x*.

$$\begin{aligned} x - y &= 6 \quad (1) \\ x - (-2) &= 6 \\ x + 2 &= 6 \\ x &= 4 \end{aligned}$$

Check (4, –2, 1) in each of the original equations to verify that it is the solution set.

CLASSROOM EXAMPLE 3 — **Solving an Inconsistent System with Three Variables**

Solve the system.

$$\begin{aligned} 3x - 5y + 2z &= 1 \quad (1) \\ 5x + 8y - z &= 4 \quad (2) \\ -6x + 10y - 4z &= 5 \quad (3) \end{aligned}$$

Solution:

Multiply equation (1) by 2 and add the result to equation (3).

$$\begin{aligned} 6x - 10y + 4z &= 2 \quad (1)\times 2 \\ -6x + 10y - 4z &= 5 \quad (3) \\ \hline 0 &= 7 \end{aligned}$$

Since a false statement results, the system is inconsistent. The solution set is ∅.

CLASSROOM EXAMPLE 4 — **Solving a System of Dependent Equations with Three Variables**

Solve the system.

$$\begin{aligned} x - y + z &= 4 \\ -3x + 3y - 3z &= -12 \\ 2x - 2y + 2z &= 8 \end{aligned}$$

Solution:

Since equation (2) is –3 times equation (1) and equation (3) is 2 times equation (1), the three equations are dependent. All three have the same graph.

The solution set is {(*x*, *y*, *z*) | *x* – *y* + *z* = 4}.

CLASSROOM EXAMPLE 5 — **Solving Another Special System**

Solve the system.

$$\begin{aligned} 2x + 3y - z &= 8 \\ \frac{1}{2}x + \frac{3}{4}y - \frac{1}{4}z &= 2 \\ x + \frac{3}{2}y - \frac{1}{2}z &= -6 \end{aligned}$$

Solution:

Eliminate the fractions in equations (2) and (3).

$$2x + 3y - z = 8 \quad (1)$$

Multiply equation (2) by 4. $\quad 2x + 3y - z = 8 \quad (4)$

Multiply equation (3) by 2. $\quad 2x + 3y - z = -12 \quad (5)$

CLASSROOM EXAMPLE 5 — **Solving Another Special System (cont'd)**

$$\begin{aligned} 2x + 3y - z &= 8 \quad (1) \\ 2x + 3y - z &= 8 \quad (4) \\ 2x + 3y - z &= -12 \quad (5) \end{aligned}$$

Equations (1) and (4) are dependent (they have the same graph).

Equations (1) and (5) are not equivalent. Since they have the same coefficients but different constant terms, their graphs have no points in common (the planes are parallel).

Thus the system is inconsistent and the solution set is ∅.