4.4 Solving Systems of Linear Equations by Matrix Methods

Objectives

1. Define a matrix.
2. Write the augmented matrix of a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.

Objective 1

Define a matrix.

Define a matrix.

A **matrix** is an ordered array of numbers.

Columns

Rows

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix}$$

The numbers are called **elements** of the matrix.

Matrices are named according to the number of **rows** and **columns** they contain.

The number of rows followed by the number of columns give the **dimensions** of the matrix.

Define a matrix.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

2 × 2 matrix

$$\begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix}$$

4 × 3 matrix

A **square matrix** is a matrix that has the same number of rows as columns.

Write the augmented matrix of a system.

An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups.

$$x - 3y = 1$$
$$2x + y = -5$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Write the augmented matrix of a system.

Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements of any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.

Write the augmented matrix of a system.

Examples of Row Operations

Row operation 1

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}$$

Interchange row 1 and row 3.

Row operation 2

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply row 1 by 3.

Write the augmented matrix of a system.

Examples of Row Operations (continued)

Row operation 3

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply row 3 by –2; add them to the corresponding numbers in row 1.

Objective 3

Use row operations to solve a system with two equations.

Use row operations to solve a system with two equations.

Row operations can be used to rewrite a matrix until it is the matrix of a system whose solution is easy to find. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two and three equations.

A matrix written as shown above with a diagonal of ones, is said to be in **row echelon form**.

CLASSROOM EXAMPLE 1 **Using Row Operations to Solve a System with Two Variables**

Use row operations to solve the system.

$$x - 2y = 9$$
$$3x + y = 13$$

Solution:

Write the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3 & 1 & 13 \end{array}\right]$$

Use row operations to change the matrix into one that leads to a system that is easy to solve.

It is best to work by columns.

CLASSROOM EXAMPLE 1 **Using Row Operations to Solve a System with Two Variables (cont'd)**

$$x - 2y = 9$$
$$3x + y = 13$$

$-3R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3 & 1 & 13 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3+(1)(-3) & 1+(-2)(-3) & 13+(-3)(9) \end{array}\right]$$

Original number from row 2 — **–3 times the number from row 1**

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 7 & -14 \end{array}\right] \tfrac{1}{7}R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 1 & -2 \end{array}\right]$$

CLASSROOM EXAMPLE 1 — Using Row Operations to Solve a System with Two Variables (cont'd)

$$x - 2y = 9$$
$$3x + y = 13$$

The matrix gives the system

$$x - 2y = 9$$
$$y = -2$$

Substitute –2 for *y* in the first equation.

$$x - 2y = 9 \qquad x - 2(-2) = 9$$
$$y = -2 \qquad x + 4 = 9$$
$$x = 5$$

The solution set is {(5, –2)}.

Objective 4

Use row operations to solve a system with three equations.

CLASSROOM EXAMPLE 2 — Using Row Operations to Solve a System with Three Variables

Use row operations to solve the system.

$$2x - y + z = 7$$
$$x - 3y - z = 7$$
$$-x + y - 5z = -9$$

Solution:

Interchange rows 1 and 2.

$$x - 3y - z = 7$$
$$2x - y + z = 7$$
$$-x + y - 5z = -9$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 2 & -1 & 1 & 7 \\ -1 & 1 & -5 & -9 \end{array}\right]$$

CLASSROOM EXAMPLE 2 — Using Row Operations to Solve a System with Three Variables (cont'd)

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 2 & -1 & 1 & 7 \\ -1 & 1 & -5 & -9 \end{array}\right] -2R_1 + R_2 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ -1 & 1 & -5 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ -1 & 1 & -5 & -9 \end{array}\right] R_1 + R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ 0 & -2 & -6 & -2 \end{array}\right]$$

CLASSROOM EXAMPLE 2 — Using Row Operations to Solve a System with Three Variables (cont'd)

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ 0 & -2 & -6 & -2 \end{array}\right] \tfrac{1}{5}R_2 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & -2 & -6 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & -2 & -6 & -2 \end{array}\right] 2R_2 + R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & \frac{-24}{5} & \frac{-24}{5} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & \frac{-24}{5} & \frac{-24}{5} \end{array}\right] \tfrac{-5}{24}R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

CLASSROOM EXAMPLE 2 — Using Row Operations to Solve a System with Three Variables (cont'd)

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This matrix gives the system

$$x - 3y - z = 7$$
$$y + \frac{3}{5}z = -\frac{7}{5}$$
$$z = 1.$$

Substitute 1 for *z* in the second equation.

$$y + \frac{3}{5}(1) = -\frac{7}{5}$$
$$y = -2$$

CLASSROOM EXAMPLE 2 — Using Row Operations to Solve a System with Three Variables (cont'd)

Substitute −2 for *y* and1 for *z* in the first equation.

$$x - 3y - z = 7$$
$$x - 3(-2) - 1 = 7$$
$$x + 5 = 7$$
$$x = 2$$

The solution set is {(2, – 2, 1)}.

Objective 5

Use row operations to solve special systems.

CLASSROOM EXAMPLE 3 — Recognizing Inconsistent Systems or Dependent Equations

Use row operations to solve the system.

$$x - y = 2$$
$$-2x + 2y = 2$$

Solution:

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & 2 \end{array}\right] 2R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 6 \end{array}\right]$$

CLASSROOM EXAMPLE 3 — Recognizing Inconsistent Systems or Dependent Equations (cont'd)

The matrix gives the system $x - y = 2$

$$0 = 6.$$

The false statement indicates that the system is inconsistent and has no solution.

The solution set is ∅.

CLASSROOM EXAMPLE 3 — Recognizing Inconsistent Systems or Dependent Equations (cont'd)

Use row operations to solve the system.

$$x - y = 2$$
$$-2x + 2y = -4$$

Solution:

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right] 2R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

CLASSROOM EXAMPLE 3 — Recognizing Inconsistent Systems or Dependent Equations (cont'd)

The matrix gives the system $x - y = 2$

$$0 = 0.$$

The true statement indicates that the system has dependent equations.

The solution set is {(*x*, *y*)| *x* – *y* = 2}.