4.1 Systems of Linear Equations in Two Variables

Objectives

1 Decide whether an ordered pair is a solution of a linear system.

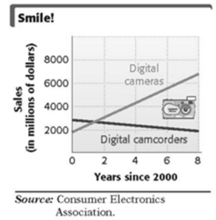
2 Solve linear systems by graphing.

3 Solve linear systems (with two equations and two variables) by substitution.

4 Solve linear systems (with two equations and two variables) by elimination.

5 Solve special systems.

Systems of Linear Equations in Two Variables



We can use linear equations to model the graphs of real applications. In this case, the sales of digital cameras and camcorders.

The two straight-line graphs intersect at the point in time when the two products had the ***same*** sales.

(Here, $x = 0$ represents 2000, $x = 1$ represents 2001, and so on; y represents sales in millions of dollars.)

A set of equations is called a **system of equations,** in this case, a **linear system of equations.**

Decide whether an ordered pair is a solution of a linear system.

The **solution set of a linear system** of equations contains all ordered pairs that satisfy all the equations of the system ***at the same time.***

CLASSROOM EXAMPLE 1 **Deciding Whether an Ordered Pair is a Solution**

Is the ordered pair a solution of the given system?

(–4, 2)

$$2x + y = -6$$
$$x + 3y = 2$$

Solution:

Replace x with –4 and y with 2 in each equation of the system.

$2x + y = -6$	$x + 3y = 2$
$2(-4) + 2 = -6$	$-4 + 3(2) = 2$
$-8 + 2 = -6$	$-4 + 6 = 2$
$-6 = -6$	$2 = 2$
True	**True**

Since (–4, 2) makes both equations true, it is a solution.

CLASSROOM EXAMPLE 1 **Deciding Whether an Ordered Pair is a Solution (cont'd)**

Is the ordered pair a solution of the given system?

(3, –12)

$$2x + y = -6$$
$$x + 3y = 2$$

Solution:

Replace x with 3 and y with −12 in each equation of the system.

$2x + y = -6$	$x + 3y = 2$
$2(3) + (-12) = -6$	$3 + 3(-12) = 2$
$6 - 12 = -6$	$3 - 36 = 2$
$-6 = -6$	$-33 = 2$
True	**False**

The ordered pair (3, –12) is not a solution of the system, since it does not make ***both*** equations true.

CLASSROOM EXAMPLE 2 **Solving a System by Graphing**

Solve the system of equations by graphing.

$$2x + y = -5$$
$$-x + 3y = 6$$

Solution:

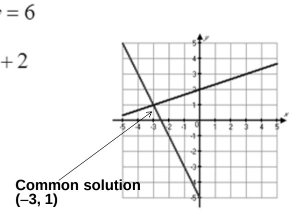
Graph each linear equation.

$2x + y = -5$

$y = -2x - 5$

$-x + 3y = 6$

$y = \frac{1}{3}x + 2$



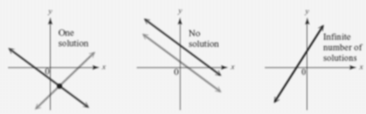
The graph suggests that the point of intersection is the ordered pair (–3, 1).

Check the solution in ***both*** equations.

Solve linear systems by graphing.

Graphs of Linear Systems in Two Variables

1. **The two graphs intersect in a single point.** The coordinates of this point give the only solution of the system. Since the system has a solution, it is **consistent.** The equations are *not* equivalent, so they are **independent.** See Figure 3(a).
2. **The graphs are parallel lines.** There is no solution common to both equations, so the solution set is $\emptyset$ and the system is **inconsistent.** Since the equations are *not* equivalent, they are **independent.** See Figure 3(b).
3. **The graphs are the same line.** Since any solution of one equation of the system is a solution of the other, the solution set is an infinite set of ordered pairs representing the points on the line. This type of system is **consistent** because there is a solution. The equations are equivalent, so they are **dependent.** See Figure 3(c).



Consistent system; independent equations (a) | Inconsistent system; independent equations (b) | Consistent system; dependent equations (c)

FIGURE 3

Solve linear systems (with two equations and two variables) by substitution.

It can be difficult to read exact coordinates, especially if they are not integers, from a graph. For this reason, we usually use algebraic methods to solve systems.

The **substitution method,** is most useful for solving linear systems in which one equation is solved or can easily be solved for one variable in terms of the other.

CLASSROOM EXAMPLE 3 **Solving a System by Substitution**

Solve the system. $5x - 3y = -6 \quad (1)$

Solution: $x = 2 - y \quad (2)$

Since equation (2) is solved for *x*, substitute 2 – *y* for *x* in equation (1).

$5x - 3y = -6 \quad (1)$

$5(2 - y) - 3y = -6$

$10 - 5y - 3y = -6$ **Distributive property**

$10 - 8y = -6$ **Combine like terms.**

$-8y = -16$ **Subtract 10.**

$y = 2$ **Divide by – 8.**

Be sure to use parentheses here.

CLASSROOM EXAMPLE 3 **Solving a System by Substitution (cont'd)**

$5x - 3y = -6 \quad (1)$

$x = 2 - y \quad (2)$

We found *y*. Now find *x* by substituting 2 for *y* in equation (2).

$$x = 2 - y$$
$$= 2 - 2 = 0$$

Thus *x* = 0 and *y* = 2, giving the ordered pair (0, 2). Check this solution in both equations of the original system.

Check:

$5x - 3y = -6 \quad (1)$ | $x = 2 - y \quad (2)$

$5(0) - 3(2) = -6$ | $0 = 2 - 2$

$0 - 6 = -6$ | $0 = 0$

$-6 = -6$ | **True**

True

The solution set is (0, 2).

Solve linear systems (with two equations and two variables) by substitution.

Solving a Linear System by Substitution

Step 1 **Solve one of the equations for either variable.** If one of the equations has a variable term with coefficient 1 or –1, choose it, since the substitution method is usually easier this way.

Step 2 **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

Step 3 **Solve** the equation from ***Step 2.***

Step 4 **Find the other value**. Substitute the result from ***Step 3*** into the equation from ***Step 1*** to find the value of the other variable.

Step 5 **Check** the solution in ***both*** of the ***original*** equations. Then write the solution set.

CLASSROOM EXAMPLE 4 **Solving a System by Substitution**

Solve the system. $4x + y = 5 \quad (1)$

Solution: $2x - 3y = 13 \quad (2)$

Step 1 **Solve one of the equations for either *x* or *y*.**

$4x + y = 5 \quad (1)$

$y = 5 - 4x$ **Subtract 4*x*.**

Step 2 **Substitute** 5 – 4*x* into equation for *y* (2).

$$2x - 3(5 - 4x) = 13$$

Step 3 **Solve.**

$$2x - 15 + 12x = 13$$
$$14x = 28$$
$$x = 2$$

CLASSROOM EXAMPLE 4 — **Solving a System by Substitution (cont'd)**

Step 4 **Now find *y*.**

$$4x + y = 5 \quad (1)$$
$$2x - 3y = 13 \quad (2)$$

$$y = 5 - 4x$$
$$y = 5 - 4(2) = -3$$

Step 5 **Check** the solution (2, –3) in both equations.

(1)	(2)
$4x + y = 5$	$2x - 3y = 13$
$4(2) + (-3) = 5$	$2(2) - 3(-3) = 13$
$8 - 3 = 5$	$4 + 9 = 13$
$5 = 5$	$13 = 13$
True	**True**

The solution set is (2, –3).

CLASSROOM EXAMPLE 5 — **Solving a System with Fractional Coefficients**

Solve the system.

$$-2x + 5y = 22 \quad (1)$$
$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{2} \quad (2)$$

Solution:

Clear the fractions in equation (2). Multiply by the LCD, 4.

$$4\left(\frac{1}{2}x + \frac{1}{4}y\right) = 4\left(\frac{1}{2}\right)$$
$$4 \cdot \frac{1}{2}x + 4 \cdot \frac{1}{4}y = 4 \cdot \frac{1}{2}$$
$$2x + y = 2 \quad (3)$$

Solve equation (3) for *y*.

$$2x + y = 2 \quad (3)$$
$$y = 2 - 2x$$

CLASSROOM EXAMPLE 5 — **Solving a System with Fractional Coefficients (cont'd)**

$$-2x + 5y = 22 \quad (1)$$
$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{2} \quad (2)$$

Substitute *y* = 2 – 2*x* for *y* in equation (1).

$$-2x + 5y = 22$$
$$-2x + 5(2 - 2x) = 22$$
$$-2x + 10 - 10x = 22$$
$$-12x + 10 = 22$$
$$-12x = 12$$
$$x = -1$$

Solve *y*.

$$y = 2 - 2x$$
$$y = 2 - 2(-1)$$
$$y = 2 + 2 = 4$$

A check verifies that the solution set is {(– 1, 4)}.

CLASSROOM EXAMPLE 6 — **Solving a System by Elimination**

Solve the system.

$$-2x + 3y = -10 \quad (1)$$
$$2x + 2y = 5 \quad (2)$$

Solution:

Adding the equations together will eliminate *x*.

$$\begin{array}{rl} -2x + 3y = -10 & (1) \\ 2x + 2y = 5 & (2) \\ \hline 5y = -5 & \\ y = -1 & \end{array}$$

To find *x*, substitute –1 for *y* in either equation.

$$2x + 2y = 5 \quad (2)$$
$$2x + 2(-1) = 5$$
$$2x - 2 = 5$$
$$2x = 7 \qquad x = \frac{7}{2}$$

The solution set is $\left\{\left(\frac{7}{2}, -1\right)\right\}$.

Solve linear systems (with two equations and two variables) by elimination.

Solving a Linear System by Elimination

Step 1 **Write both equations in standard form *Ax + By = C*.**

Step 2 **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the *x*- or *y*-terms is 0.

Step 3 **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4 **Solve** the equation from ***Step 3*** for the remaining variable.

Step 5 **Find the other value.** Substitute the result of ***Step 4*** into either of the original equations and solve for the other variable.

Step 6 **Check** the ordered-pair solution in ***both*** of the ***original*** equations. Then write the solution set.

CLASSROOM EXAMPLE 7 — **Solving a System by Elimination**

Solve the system.

$$2x + 3y = 19 \quad (1)$$
$$3x - 7y = -6 \quad (2)$$

Solution:

Step 1 **Both equations are in standard form.**

Step 2 **Select a variable to eliminate,** say *y*. Multiply equation (1) by 7 and equation (2) by 3.

Step 3 **Add.**

$$\begin{array}{r} 14x + 21y = 133 \\ 9x - 21y = -18 \\ \hline 23x = 115 \end{array}$$

Step 4 **Solve** for *x*.

$$x = 5$$

CLASSROOM EXAMPLE 7 **Solving a System by Elimination (cont'd)**

$$2x + 3y = 19 \quad (1)$$
$$3x - 7y = -6 \quad (2)$$

Step 5 **To find *y* substitute** 5 for *x* in either equation (1) or equation (2).

$$2x + 3y = 19 \quad (1)$$
$$2(5) + 3y = 19$$
$$10 + 3y = 19$$
$$3y = 9$$
$$y = 3$$

Step 6 **To check** substitute 5 for *x* and 3 for *y* in both equations (1) and (2).

The ordered pair checks, the solution set is {(5, 3)}.

CLASSROOM EXAMPLE 8 **Solving a System of Dependent Equations**

Solve the system.
$$2x + y = 6 \quad (1)$$
Solution:
$$-8x - 4y = -24 \quad (2)$$

Multiply equation (1) by 4 and add the result to equation (2).

$$8x + 4y = 24 \quad (1)$$
$$-8x - 4y = -24 \quad (2)$$
$$0 = 0 \quad \textbf{True}$$

Equations (1) and (2) are equivalent and have the same graph. The equations are dependent.

The solution set is the set of all points on the line with equation 2*x* + *y* = 6, written in set-builder notation {(*x*, *y*) | 2*x* + *y* = 6}.

CLASSROOM EXAMPLE 9 **Solving an Inconsistent System**

Solve the system.
$$4x - 3y = 8 \quad (1)$$
Solution:
$$8x - 6y = 14 \quad (2)$$

Multiply equation (1) by –2 and add the result to equation (2).

$$-8x + 6y = -16 \quad (1)$$
$$8x - 6y = 14 \quad (2)$$
$$0 = -2 \quad \textbf{False}$$

The result of adding the equations is a false statement, which indicates the system is inconsistent. The graphs would be parallel lines. There are no ordered pairs that satisfy both equations.

The solution set is ∅.

Solving special systems.

Special Cases of Linear Systems

If both variables are eliminated when a system of linear equations is solved,

1. there are infinitely many solutions if the resulting statement is ***true*;**

2. there is no solution if the resulting statement is ***false*.**

CLASSROOM EXAMPLE 10 **Using Slope-Intercept Form to Determine the Number of Solutions**

Write each equation in slope-intercept form and then tell how many solutions the system has.

$$3x - 6y = 9 \quad (1)$$
$$x - 2y = 3 \quad (2)$$

Solution:
Rewrite both equations in y-intercept form.

$$3x - 6y = 9 \quad (1)$$
$$-6y = -3x + 9$$
$$\frac{-6y}{3} = \frac{-3x + 9}{3}$$
$$-2y = -x + 3$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

$$x - 2y = 3 \quad (2)$$
$$-2y = -x + 3$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

Both lines have the same slope and same y-intercept. They coincide and therefore have infinitely many solutions.

CLASSROOM EXAMPLE 10 **Using Slope-Intercept Form to Determine the Number of Solutions (cont'd)**

Write each equation in slope-intercept form and then tell how many solutions the system has.

$$-2x = 5y + 1 \quad (1)$$
$$-4x = 10y + 3 \quad (2)$$

Solution:
Rewrite both equations in y-intercept form.

$$-2x = 5y + 1 \quad (1)$$
$$-2x - 1 = 5y$$
$$-\frac{2}{5}x - \frac{1}{5} = y$$
$$y = -\frac{2}{5}x - \frac{1}{5}$$

$$-4x = 10y + 3 \quad (2)$$
$$-4x - 3 = 10y$$
$$-\frac{4}{10}x - \frac{3}{10} = y$$
$$y = -\frac{2}{5}x - \frac{3}{10}$$

Both lines have the same slope, but different y-intercepts. They are parallel and therefore have no solutions.

4.2 Systems of Linear Equations in Three Variables

Objectives

1. Understand the geometry of systems of three equations in three variables.
2. Solve linear systems (with three equations and three variables) by elimination.
3. Solve linear systems (with three equations and three variables) in which some of the equations have missing terms.
4. Solve special systems.

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Systems of Linear Equations in Three Variables

A solution of an equation in three variables, such as $2x + 3y - z = 4$ is called an **ordered triple** and is written (x, y, z).

Understand the geometry of system of three equations in three variables.

The graph of a linear equation with three variables is a ***plane,*** not a line.

A number of possible solutions are shown below.

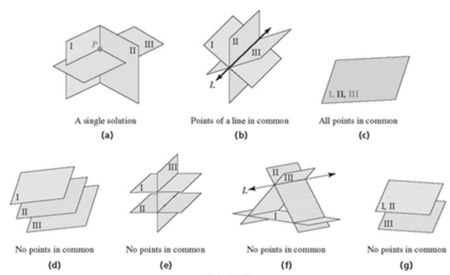

FIGURE 7

Understand the geometry of system of three equations in three variables.

Graphs of Linear Systems in Three Variables

1. **The three planes may meet at a single, common point** that is the solution of the system. (*See* **Figure 7a**).
2. **The three planes may have the points of a line in common,** so that the infinite set of points that satisfy the equation of the line is the solution of the system. (*See* **Figure 7b**).
3. **The three planes may coincide,** so that the solution of the system is the set of all points on a plane. (*See* **Figure 7c**).
4. **The planes may have no points common to all three,** so that there is no solution of the system. (*See* **Figures 7d-g**).

Sole linear systems (with three equations and three variables) by elimination.

In the steps that follow, we use the term **focus variable** to identify the first variable to be eliminated in the process. The focus variable will always be present in the **working equation,** which will be used twice to eliminate this variable.

Solve linear systems (with three equations and three variables) by elimination.

Solving a Linear System in Three Variables

Step 1 **Select a variable and an equation.** A good choice for the variable, which we call the ***focus variable,*** is one that has coefficient 1 or −1. Then select an equation, one that contains the focus variable, as the ***working equation.***

Step 2 **Eliminate the focus variable.** Use the working equation and one of the other two equations of the original system. The result is an equation in two variables.

Step 3 **Eliminate the focus variable again.** Use the working equation and the remaining equation of the original system. The result is another equation in two variables.

Solve linear systems (with three equations and three variables) by elimination.

Solving a Linear System in Three Variables (cont'd)

Step 4 **Write the equations in two variables that result from *Steps 2 and 3* as a system, and solve it.** Doing this gives the values of two of the variables.

Step 5 **Find the value of the remaining variable.** Substitute the values of the two variables found in ***Step 4*** into the working equation to obtain the value of the focus variable.

Step 6 **Check** the ordered-pair solution in ***each*** of the ***original*** equations of the system. Then write the solution set.

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables**

Solve the system.

$$x + y + z = 2 \quad (1)$$
$$x - y + 2z = 2 \quad (2)$$
$$-x + 2y - z = 1 \quad (3)$$

Solution:

Step 1 Select the variable *y* as the focus variable and equation (1) as the working equation.

$$x + y + z = 2 \quad (1)$$

Step 2 Multiplying can be skipped as the focus variable can be eliminated when adding equations (1) and (2)

$$\begin{aligned} x + y + z &= 2 \quad (1)\\ x - y + 2z &= 2 \quad (2)\\ \hline 2x \quad + 3z &= 4 \quad (4) \end{aligned}$$

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

$$x + y + z = 2 \quad (1)$$
$$x - y + 2z = 2 \quad (2)$$
$$-x + 2y - z = 1 \quad (3)$$

Step 3 Use the working equation (1), multiply both sides by −2 and add to equation (3) to again eliminate focus variable *y*.

$$\begin{aligned} -2x - 2y - 2z &= -4 \quad (1)\\ -x + 2y - z &= 1 \quad (3)\\ \hline -3x \quad - 3z &= -3 \quad (5) \end{aligned}$$

Make sure equation (5) has the same variables as equation 4.

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

$$x + y + z = 2 \quad (1)$$
$$x - y + 2z = 2 \quad (2)$$
$$-x + 2y - z = 1 \quad (3)$$

Step 4 Write the equations in two variables that result in ***Steps 2 and 3*** as a system, then solve to eliminate *z*. Substitute the value of *x* into equation (4) to sole for *z*.

$$\begin{aligned} 2x + 3z &= 4 \quad (4)\\ -3x - 3z &= -3 \quad (5)\\ \hline -x &= 1 \quad \text{or} \quad x = -1 \end{aligned}$$

$$\begin{aligned} 2(-1) + 3z &= 4 \quad (4)\\ -2 + 3z &= 4\\ 3z &= 6\\ z &= 2 \end{aligned}$$

Step 5 Substitute –1 for *x* and 2 for *z* in equation (1) to find *y*.

$$\begin{aligned} -1 + y + 2 &= 2 \quad (1)\\ y + 1 &= 2\\ y &= 1 \end{aligned}$$

CLASSROOM EXAMPLE 1 **Solving a System in Three Variables (cont'd)**

Step 6 Check the ordered triple (–1, 1, 2) to be sure the solution satisfies all three equations in the original system.

$$x + y + z = 2 \quad (1) \qquad -1 + 1 + 2 = 2$$
$$x - y + 2z = 2 \quad (2) \qquad -1 - 1 + 2(2) = 2$$
$$-x + 2y - z = 1 \quad (3) \qquad -(-1) + 2(1) - (2) = 1$$

The solution set is {(–1, 1, 2)}.

Write the values of *x, y,* and *z* in the correct order.

CLASSROOM EXAMPLE 2 **Solving a System of Equations with Missing Terms**

Solve the system.

Solution:

$$x - y = 6 \quad (1)$$
$$2y + 5z = 1 \quad (2)$$
$$3x - 4z = 8 \quad (3)$$

Since equation (3) is missing *y*, eliminate *y* again from equations (1) and (2). Multiply equation (1) by 2 and add the result to equation (2).

$$x - y = 6 \quad (1) \qquad 2x - 2y = 12 \quad (1)\times 2$$
$$2y + 5z = 1 \quad (2) \qquad 2y + 5z = 1 \quad (2)$$
$$2x + 5z = 13 \quad (4)$$

CLASSROOM EXAMPLE 2 Solving a System of Equations with Missing Terms (cont'd)

Use equation (4) together with equation (3) to eliminate *x*. Multiply equation (4) by 3 and equation (3) by –2. Then add the results.

$$\begin{array}{llll} 2x \quad + 5z = 13 & (4)\times(3) & 6x + 15z = 39 \\ 3x - \quad 4z = 8 & (3)\times(-2) & -6x + 8z = -16 \\ & & \overline{\quad 23z = 23} \\ & & z = 1 \end{array}$$

CLASSROOM EXAMPLE 2 Solving a System of Equations with Missing Terms (cont'd)

Substitute 1 for *z* in equation (2) to find *y*.

$$\begin{aligned} 2y + 5z &= 1 \quad (2) \\ 2y + 5(1) &= 1 \\ 2y + 5 &= 1 \\ 2y &= -4 \\ y &= -2 \end{aligned}$$

Substitute –2 for *y* in (1) to find *x*.

$$\begin{aligned} x - y &= 6 \quad (1) \\ x - (-2) &= 6 \\ x + 2 &= 6 \\ x &= 4 \end{aligned}$$

Check (4, –2, 1) in each of the original equations to verify that it is the solution set.

CLASSROOM EXAMPLE 3 Solving an Inconsistent System with Three Variables

Solve the system.

$$\begin{aligned} 3x - 5y + 2z &= 1 \quad (1) \\ 5x + 8y - z &= 4 \quad (2) \\ -6x + 10y - 4z &= 5 \quad (3) \end{aligned}$$

Solution:

Multiply equation (1) by 2 and add the result to equation (3).

$$\begin{array}{ll} 6x - 10y + 4z = 2 & (1)\times 2 \\ -6x + 10y - 4z = 5 & (3) \\ \hline 0 = 7 & \end{array}$$

Since a false statement results, the system is inconsistent. The solution set is ∅.

CLASSROOM EXAMPLE 4 Solving a System of Dependent Equations with Three Variables

Solve the system.

$$\begin{aligned} x - y + z &= 4 \\ -3x + 3y - 3z &= -12 \\ 2x - 2y + 2z &= 8 \end{aligned}$$

Solution:

Since equation (2) is –3 times equation (1) and equation (3) is 2 times equation (1), the three equations are dependent. All three have the same graph.

The solution set is {(*x*, *y*, *z*) | *x* – *y* + *z* = 4}.

CLASSROOM EXAMPLE 5 Solving Another Special System

Solve the system.

$$\begin{aligned} 2x + 3y - z &= 8 \\ \frac{1}{2}x + \frac{3}{4}y - \frac{1}{4}z &= 2 \\ x + \frac{3}{2}y - \frac{1}{2}z &= -6 \end{aligned}$$

Solution:

Eliminate the fractions in equations (2) and (3).

$$2x + 3y - z = 8 \quad (1)$$

Multiply equation (2) by 4. $2x + 3y - z = 8 \quad (4)$

Multiply equation (3) by 2. $2x + 3y - z = -12 \quad (5)$

CLASSROOM EXAMPLE 5 Solving Another Special System (cont'd)

$$\begin{aligned} 2x + 3y - z &= 8 \quad (1) \\ 2x + 3y - z &= 8 \quad (4) \\ 2x + 3y - z &= -12 \quad (5) \end{aligned}$$

Equations (1) and (4) are dependent (they have the same graph).

Equations (1) and (5) are not equivalent. Since they have the same coefficients but different constant terms, their graphs have no points in common (the planes are parallel).

Thus the system is inconsistent and the solution set is ∅.

4.3 Applications of Systems of Linear Equations

Objectives

1 Solve geometry problems by using two variables.

2 Solve money problems by using two variables.

3 Solve mixture problems by using two variables.

4 Solve distance–rate–time problems by using two variables.

5 Solve problems with three variables by using a system of three equations.

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Applications of Systems of Linear Equations

PROBLEM-SOLVING HINT

When solving an applied problem using two variables, it is a good idea to pick letters that correspond to the descriptions of the unknown quantities. For example above, we could choose *c* to represent the number of citrons and *w* to represent the number of wood apples.

Applications of Systems of Linear Equations

Solving an Applied Problem by Writing a System of Equations

Step 1 **Read** the problem, several times if necessary. What information is given? What is to be found? This is often stated in the last sentence.

Step 2 **Assign variables** to represent the unknown values. Use a sketch, diagram, or table, as needed.

Step 3 **Write a system of equations** using the variable expressions.

Step 4 **Solve** the system of equations.

Step 5 **State the answer** to the problem. Label it appropriately. Does it seem reasonable?

Step 6 **Check** the answer in the words of the ***original*** problem.

CLASSROOM EXAMPLE 1 **Finding the Dimensions of a Soccer Field**

A rectangular soccer field has perimeter 360 yd. Its length is 20 yd more than its width. What are its dimensions?

Solution:

Step 1 **Read** the problem again. We are asked to find the dimensions of the field.

Step 2 **Assign variables.**

Let L = the length and W = the width.

Step 3 **Write a system of equations.**

The perimeter of a rectangle is given by $2W + 2L = 360$.

Since the length is 20 yd more than the width, $L = W + 20$.

CLASSROOM EXAMPLE 1 **Finding the Dimensions of a Soccer Field (cont'd)**

The system is

$$L = W + 20 \quad (1)$$
$$2W + 2L = 360 \quad (2)$$

Step 4 **Solve.** Substitute $W + 20$ for L in equations (2).

$$2W + 2(W + 20) = 360$$
$$2W + 2W + 40 = 360$$
$$4W = 320$$
$$W = 80$$

Substitute $W = 80$ into equation (1). $L = 80 + 20 = 100$

CLASSROOM EXAMPLE 1 **Finding the Dimensions of a Soccer Field (cont'd)**

Step 5 **State the answer.**

The length of the field is 100 yards and the width is 80 yards.

Step 6 **Check.**

The perimeter of the soccer field is 2(100) + 2(80) = 360 yd, and the length, 100 yards is 20 more than the width, since 100 – 20 = 80.

The answer is correct.

CLASSROOM EXAMPLE 2 | **Solving a Problem about Ticket Prices**

For the 2009 Major League Baseball and National Football League seasons, based on average ticket prices, three baseball tickets and two football tickets would have cost $229.90. Two baseball tickets and one football ticket would have cost $128.27. What were the average ticket prices for the tickets for the two sports? (*Source:* Team Marketing Report.)

Solution:

Step 1 **Read** the problem again. There are two unknowns.

Step 2 **Assign variables.**

Let x = the average cost of baseball tickets, and y = the average cost of football tickets.

CLASSROOM EXAMPLE 2 | **Solving a Problem about Ticket Prices (cont'd)**

Step 3 **Write a system of equations.**

$$3x + 2y = 229.90 \quad (1)$$
$$2x + y = 128.27 \quad (2)$$

Step 4 **Solve.**

Multiply equation (2) by –2 and add to equation (1).

$$\begin{array}{rcll} 3x + 2y & = & 229.90 & (1) \\ -4x - 2y & = & -256.54 & -2 \times (2) \\ \hline -x & = & -26.64 & \\ x & = & 26.64 & \end{array}$$

Let x = 26,64 in equation (2).

$$2(26.64) + y = 128.27$$
$$53.28 + y = 128.27$$
$$y = 74.99$$

CLASSROOM EXAMPLE 2 | **Solving a Problem about Ticket Prices (cont'd)**

Step 5 **State the answer.**

The average cost of a baseball ticket is $26.64 and the average cost of a football ticket is $74.99.

Step 6 **Check.**

3(26.64) + 2(74.99) = 229.90

and 2(26.64) + 74.99 = 128.27.

The answer is correct.

CLASSROOM EXAMPLE 3 | **Solving a Mixture Problem**

A grocer has some $4-per-lb coffee and some $8-per-lb coffee that she will mix to make 50 lb of $5.60-per-lb coffee. How many pounds of each should be used?

Solution:

Step 1 **Read** the problem.

Step 2 **Assign variables.**

Let x = number of pounds of the $4-per-lb coffee and y = the number of pounds of the $8-per-pound coffee.

Price per Pound	Number of Pounds	Value of Coffee
$4	x	$4x$
$8	y	$8y$
$5.60	50	5.6(50) = 280

CLASSROOM EXAMPLE 3 | **Solving a Mixture Problem (cont'd)**

Step 3 **Write a system of equations.**

$$x + y = 50 \quad (1)$$
$$4x + 8y = 280 \quad (2)$$

Step 4 **Solve.**

To eliminate x, multiply equation (1) by –4 and add to equation (2).

$$\begin{array}{rcll} -4x - 4y & = & -200 & -4 \times (1) \\ 4x + 8y & = & 280 & (2) \\ \hline 4y & = & 80 & \\ y & = & 20 & \end{array}$$

Since y = 20 and $x + y$ = 50, x = 30.

CLASSROOM EXAMPLE 3 | **Solving a Mixture Problem (cont'd)**

Step 5 **State the answer.**

To mix the coffee, 30 lb of $4-per-lb coffee and 20 lb of $8-per-lb coffee should be used.

Step 6 **Check.**

30 + 20 = 50

and 4(30) + 8(20) = 280.

The answer is correct.

CLASSROOM EXAMPLE 4 Solving a Motion Problem

A train travels 600 mi in the same time that a truck travels 520 mi. Find the speed of each vehicle if the train's average speed is 8 mph faster than the truck's.

Solution:

Step 1 **Read** the problem.

We need to find the speed of each vehicle.

Step 2 **Assign variables.**

Let x = the train's speed and y = the truck's speed.

	Distance	Rate	Time
Train	600	x	$600/x$
Truck	520	y	$520/y$

The times must be equal.

CLASSROOM EXAMPLE 4 Solving a Motion Problem (cont'd)

Step 3 **Write a system of equations.**

$$\frac{600}{x} = \frac{520}{y}$$
$$600y = 520x$$
$$-520x + 600y = 0 \quad (1)$$
$$x = y + 8. \quad (2)$$

Step 4 **Solve.**

Substitute $y + 8$ for x in equation (1) to find y.

$$-520x + 600y = 0 \quad (1)$$
$$-520(y + 8) + 600y = 0$$
$$-520y - 4160 + 600y = 0$$
$$80y = 4160$$
$$y = 52$$

Since $y = 52$ and $x = y + 8$, $x = 60$.

CLASSROOM EXAMPLE 4 Solving a Motion Problem (cont'd)

Step 5 **State the answer.**

The train's speed is 60 mph, the truck's speed is 52 mph.

Step 6 **Check.**

$$60 = 52 + 8$$

It would take the train 10 hours to travel 600 miles at 60 mph, which is the same amount of time it would take the truck to travel 520 miles at 52 mph.

The answer is correct.

Objective 5

Solve problems with three variables by using a system of three equations.

Solve problems with three variables by using a system of three equations.

PROBLEM-SOLVING HINT

If an application requires finding ***three*** unknown quantities, we can use a system of ***three*** equations to solve it. We extend the method used for two unknowns.

CLASSROOM EXAMPLE 5 Solving a Problem Involving Prices

A department store display features three kinds of perfume: Felice, Vivid, and Joy. There are 10 more bottles of Felice than Vivid, and 3 fewer bottles of Joy than Vivid. Each bottle of Felice costs $8, Vivid costs $15, and Joy costs $32. The total value of the all the perfume is $589. How many bottles of each are there?

Solution:

Step 1 **Read** the problem.

There are 3 unknowns.

Step 2 **Assign variables.**

Let x = the number of bottles of Felice at $8
y = the number of bottles of Vivid at $15, and
z = the number of bottles of Joy at $32.

CLASSROOM EXAMPLE 5 Solving a Problem Involving Prices (cont'd)

Step 3 **Write a system of equations.**

There are 10 more bottles of Felice, so $x = y + 10$. (1)

There are 3 fewer bottles of Joy than Vivid, so $z = y - 3$. (2)

The total value is $589, so $8x + 15y + 32z = 589$. (3)

CLASSROOM EXAMPLE 5 Solving a Problem Involving Prices (cont'd)

Step 4 **Solve.**

Substitute $y + 10$ for x and $y - 3$ for z in equation (3) to find y.

$$8(y + 10) + 15y + 32(y - 3) = 589$$
$$8y + 80 + 15y + 32y - 96 = 589$$
$$55y - 16 = 589$$
$$55y = 605$$
$$y = 11$$

Since $y = 11$, $x = y + 10 = 21$ and $z = y - 3 = 8$.

CLASSROOM EXAMPLE 5 Solving a Problem Involving Prices (cont'd)

Step 5 **State the answer.**

There are 21 bottles Felice, 11 of Vivid, and 8 of Joy.

Step 6 **Check.**

$$21(8) + 11(15) + 8(32) = 589$$

The answer is correct.

CLASSROOM EXAMPLE 6 Solving a Business Production Problem

A paper mill makes newsprint, bond, and copy machine paper.

- Each ton of newsprint requires 3 tons of recycled paper and 1 ton of wood pulp.
- Each ton on bond requires 2 tons of recycled paper, and 4 tons of wood pulp, and 3 tons of rags.
- Each ton of copy machine paper requires 2 tons of recycled paper, 3 tons of wood pulp, and 2 tons of rags.

The mill has 4200 tons of recycled paper, 5800 tons of wood pulp, and 3900 tons of rags. How much of each kind of paper can be made from these supplies?

Solution:

Step 1 **Read** the problem.

CLASSROOM EXAMPLE 6 Solving a Business Production Problem (cont'd)

Step 2 **Assign variables.**

Let x = the number of tons of newsprint
y = the number of tons of bond, and
z = the number of tons of copy machine paper.

Step 3 **Write a system of equations.**

$$3x + 2y + 2z = 4200 \quad (1)$$
$$x + 4y + 3z = 5800 \quad (2)$$
$$3y + 2z = 3900 \quad (3)$$

CLASSROOM EXAMPLE 6 Solving a Business Production Problem (cont'd)

Step 4 **Solve** the system to find $x = 400$, $y = 900$, and $z = 600$.

$$3x + 2y + 2z = 4200$$
$$x + 4y + 3z = 5800$$
$$3y + 2z = 3900$$

Step 5 **State the answer.**

The paper mill can make 400 tons of newsprint, 900 tons of bond, and 600 tons of copy machine paper.

Step 6 **Check** that these values satisfy the conditions of the problem.

The answer is correct.

4.4 Solving Systems of Linear Equations by Matrix Methods

Objectives

1. Define a matrix.
2. Write the augmented matrix of a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.

Objective 1

Define a matrix.

Define a matrix.

A **matrix** is an ordered array of numbers.

Columns

Rows

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix}$$

The numbers are called **elements** of the matrix.

Matrices are named according to the number of **rows** and **columns** they contain.

The number of rows followed by the number of columns give the **dimensions** of the matrix.

Define a matrix.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

2 × 2 matrix

$$\begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix}$$

4 × 3 matrix

A **square matrix** is a matrix that has the same number of rows as columns.

Write the augmented matrix of a system.

An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups.

$$x - 3y = 1$$
$$2x + y = -5$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Write the augmented matrix of a system.

Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements of any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.

Write the augmented matrix of a system.

Examples of Row Operations

Row operation 1

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}$$

Interchange row 1 and row 3.

Row operation 2

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply row 1 by 3.

Write the augmented matrix of a system.

Examples of Row Operations (continued)

Row operation 3

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}$$

Multiply row 3 by –2; add them to the corresponding numbers in row 1.

Objective 3

Use row operations to solve a system with two equations.

Use row operations to solve a system with two equations.

Row operations can be used to rewrite a matrix until it is the matrix of a system whose solution is easy to find. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two and three equations.

A matrix written as shown above with a diagonal of ones, is said to be in **row echelon form**.

CLASSROOM EXAMPLE 1 **Using Row Operations to Solve a System with Two Variables**

Use row operations to solve the system.

$$x - 2y = 9$$
$$3x + y = 13$$

Solution:

Write the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3 & 1 & 13 \end{array}\right]$$

Use row operations to change the matrix into one that leads to a system that is easy to solve.

It is best to work by columns.

CLASSROOM EXAMPLE 1 **Using Row Operations to Solve a System with Two Variables (cont'd)**

$$x - 2y = 9$$
$$3x + y = 13$$

$-3R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3 & 1 & 13 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 3+(1)(-3) & 1+(-2)(-3) & 13+(-3)(9) \end{array}\right]$$

Original number from row 2 **–3 times the number from row 1**

$$\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 7 & -14 \end{array}\right] \tfrac{1}{7}R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 1 & -2 \end{array}\right]$$

CLASSROOM EXAMPLE 1 **Using Row Operations to Solve a System with Two Variables (cont'd)**

$$x - 2y = 9$$
$$3x + y = 13$$

The matrix gives the system

$$x - 2y = 9$$
$$y = -2$$

Substitute –2 for *y* in the first equation.

$$x - 2y = 9 \qquad x - 2(-2) = 9$$
$$y = -2 \qquad x + 4 = 9$$
$$x = 5$$

The solution set is {(5, –2)}.

Objective 4

Use row operations to solve a system with three equations.

CLASSROOM EXAMPLE 2 **Using Row Operations to Solve a System with Three Variables**

Use row operations to solve the system.

$$2x - y + z = 7$$
$$x - 3y - z = 7$$
$$-x + y - 5z = -9$$

Solution:

Interchange rows 1 and 2.

$$x - 3y - z = 7$$
$$2x - y + z = 7$$
$$-x + y - 5z = -9$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 2 & -1 & 1 & 7 \\ -1 & 1 & -5 & -9 \end{array}\right]$$

CLASSROOM EXAMPLE 2 **Using Row Operations to Solve a System with Three Variables (cont'd)**

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 2 & -1 & 1 & 7 \\ -1 & 1 & -5 & -9 \end{array}\right] -2R_1 + R_2 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ -1 & 1 & -5 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ -1 & 1 & -5 & -9 \end{array}\right] R_1 + R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ 0 & -2 & -6 & -2 \end{array}\right]$$

CLASSROOM EXAMPLE 2 **Using Row Operations to Solve a System with Three Variables (cont'd)**

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 5 & 3 & -7 \\ 0 & -2 & -6 & -2 \end{array}\right] \tfrac{1}{5}R_2 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & -2 & -6 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & -2 & -6 & -2 \end{array}\right] 2R_2 + R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & \frac{-24}{5} & \frac{-24}{5} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & \frac{-24}{5} & \frac{-24}{5} \end{array}\right] \tfrac{-5}{24}R_3 \qquad \left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

CLASSROOM EXAMPLE 2 **Using Row Operations to Solve a System with Three Variables (cont'd)**

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 7 \\ 0 & 1 & \frac{3}{5} & \frac{-7}{5} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This matrix gives the system

$$x - 3y - z = 7$$
$$y + \frac{3}{5}z = -\frac{7}{5}$$
$$z = 1.$$

Substitute 1 for *z* in the second equation.

$$y + \frac{3}{5}(1) = -\frac{7}{5}$$
$$y = -2$$

CLASSROOM EXAMPLE 2 **Using Row Operations to Solve a System with Three Variables (cont'd)**

Substitute −2 for *y* and1 for *z* in the first equation.

$$x - 3y - z = 7$$
$$x - 3(-2) - 1 = 7$$
$$x + 5 = 7$$
$$x = 2$$

The solution set is {(2, – 2, 1)}.

Objective 5

Use row operations to solve special systems.

CLASSROOM EXAMPLE 3 **Recognizing Inconsistent Systems or Dependent Equations**

Use row operations to solve the system.

$$x - y = 2$$
$$-2x + 2y = 2$$

Solution:

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & 2 \end{array}\right] 2R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 6 \end{array}\right]$$

CLASSROOM EXAMPLE 3 **Recognizing Inconsistent Systems or Dependent Equations (cont'd)**

The matrix gives the system $x - y = 2$

$$0 = 6.$$

The false statement indicates that the system is inconsistent and has no solution.

The solution set is ∅.

CLASSROOM EXAMPLE 3 **Recognizing Inconsistent Systems or Dependent Equations (cont'd)**

Use row operations to solve the system.

$$x - y = 2$$
$$-2x + 2y = -4$$

Solution:

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right] 2R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

CLASSROOM EXAMPLE 3 **Recognizing Inconsistent Systems or Dependent Equations (cont'd)**

The matrix gives the system $x - y = 2$

$$0 = 0.$$

The true statement indicates that the system has dependent equations.

The solution set is {(*x*, *y*)| *x* – *y* = 2}.