7.1 Rational Expressions and Functions; Multiplying and Dividing

Objectives

1. Define rational expressions.
2. Define rational functions and describe their domains.
3. Write rational expressions in lowest terms.
4. Multiply rational expressions.
5. Find reciprocals of rational expressions.
6. Divide rational expressions.

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Define rational expressions.

A **rational expression** or ***algebraic fraction,*** is the quotient of two polynomials, again with the denominator not 0.

For example:

$$\frac{x}{y},\quad \frac{-a}{4},\quad \frac{m+4}{m+2},\quad \frac{8x^2-2x+5}{4x^2+5x},\quad x^5\left(\text{or } \frac{x^5}{1}\right)$$

are all rational expressions. Rational expressions are elements of the set

$$\left\{\frac{P}{Q}\,\middle|\,P \text{ and } Q \text{ are polynomials, with } Q \neq 0\right\}.$$

Define rational functions and describe their domains.

A function that is defined by a quotient of polynomials is called a **rational function** and has the form

$$f(x) = \frac{P(x)}{Q(x)},\ \text{where } Q(x) \neq 0.$$

The domain of the rational function consists of all real numbers except those that make *Q*(*x*)—that is, the denominator—equal to 0.

For example, the domain of $f(x) = \dfrac{2}{\underbrace{x-5}_{\text{Cannot equal 0}}}$

includes all real numbers except 5, because 5 would make the denominator equal to 0.

CLASSROOM EXAMPLE 1 Finding Domains of Rational Functions

For each rational function, find all numbers that are not in the domain. Then give the domain in set-builder notation.

$$f(x) = \frac{x+6}{x^2-x-6} \qquad f(x) = \frac{3+2x}{5}$$

Solution:

$$x^2 - x - 6 = 0$$
$$(x + 2)(x - 3) = 0$$
$$x + 2 = 0 \text{ or } x - 3 = 0$$
$$x = -2 \text{ or } \quad x = 3$$
$$\{x \mid x \neq -2, 3\}$$

The denominator 5 cannot ever be 0, so the domain includes all real numbers. $(-\infty, \infty)$

Write rational expressions in lowest terms.

Fundamental Property of Rational Numbers

If $\frac{a}{b}$ is a rational number and if *c* is any nonzero real number, then

$$\frac{a}{b} = \frac{ac}{bc}.$$

That is, the numerator and denominator of a rational number may either be multiplied or divided by the same ***nonzero number*** without changing the value of the rational number.

A rational expression is a quotient of two polynomials. Since the value of a polynomial is a real number for every value of the variable for which it is defined, any statement that applies to rational numbers will also apply to rational expressions.

Write rational expressions in lowest terms.

Writing a Rational Expression in Lowest Terms

Step 1 **Factor** both the numerator and denominator to find their greatest common factor (GCF).

Step 2 **Apply the fundamental property.** Divide out common factors.

CAUTION ***Be careful! When using the fundamental property of rational numbers, only common factors may be divided. Remember to factor before writing a fraction in lowest terms.***

CLASSROOM EXAMPLE 2 — **Writing Rational Expressions in Lowest Terms**

Write each rational expression in lowest terms.

$$\frac{y^2+2y-3}{y^2-3y+2} \qquad \frac{y+2}{y^2+4} \qquad \frac{1+p^3}{1+p}$$

Solution:

$$=\frac{(y+3)(y-1)}{(y-2)(y-1)}$$

$$=\frac{(y+3)}{(y-2)}\cdot 1$$

$$=\frac{(y+3)}{(y-2)}$$

The denominator cannot be factored, so this expression cannot be simplified further and is in lowest terms.

$$=\frac{(1+p)(1-p+p^2)}{(1+p)}$$

$$=1-p+p^2$$

CLASSROOM EXAMPLE 3 — **Writing Rational Expressions in Lowest Terms**

Write each rational expression in lowest terms.

$$\frac{r-1}{1-r} \qquad \frac{p-2}{4-p^2}$$

Solution:

$$\frac{r-1}{1-r}=\frac{r-1}{-1(r-1)}$$

$$=\frac{1}{-1}=-1$$

$$=\frac{p-2}{-1(p^2-4)}$$

$$=\frac{p-2}{-1(p-2)(p+2)}$$

$$=\frac{1}{-1(p+2)} \qquad =\frac{-1}{p+2}$$

Write rational expressions in lowest terms.

Quotient of Opposites

In general, if the numerator and the denominator of a rational expression are opposites, then the expression equals –1.

$$\frac{q-7}{7-q}=-1 \quad \text{and} \quad \frac{-5a+2b}{5a-2b}=-1$$

Numerator and denominator in each expression are opposites.

$$\frac{r-2}{r+2}$$

Numerator and denominator are ***not*** opposites.

Multiply rational expressions.

Multiplying Rational Expressions

Step 1 **Factor** all numerators and denominators as completely as possible.

Step 2 **Apply the fundamental property.**

Step 3 **Multiply** the numerators and multiply the denominators.

Step 4 **Check** to be sure that the product is in lowest terms.

CLASSROOM EXAMPLE 4 — **Multiplying Rational Expressions**

Multiply.

$$\frac{c^2+2c}{c^2-4}\cdot\frac{c^2-4c+4}{c^2-c} \qquad \frac{m^2-16}{m+2}\cdot\frac{1}{m+4}$$

Solution:

$$=\frac{c(c+2)}{(c-2)(c+2)}\cdot\frac{(c-2)(c-2)}{c(c-1)}$$

$$=\frac{(c-2)}{(c-1)}$$

$$=\frac{(m-4)(m+4)}{m+2}\cdot\frac{1}{m+4}$$

$$=\frac{m-4}{m+2}$$

Objective 5

Find reciprocals of rational expressions.

Find reciprocals of rational expressions.

Finding the Reciprocal

To find the reciprocal of a nonzero rational expression, invert the rational expression.

Two rational expressions are reciprocals of each other if they have a product of 1. Recall that 0 has no reciprocal.

Divide rational expressions.

Dividing Rational Expressions

To divide two rational expressions, ***multiply*** the first (the ***dividend***) by the reciprocal of the second (the ***divisor***).

CLASSROOM EXAMPLE 5 **Dividing Rational Expressions**

Divide.

$$\frac{5p+2}{6} \div \frac{15p+6}{5}$$

$$\frac{q^2+2q}{5+q} \div \frac{4-q^2}{3q-6}$$

Solution:

$$= \frac{5p+2}{6} \cdot \frac{5}{15p+6}$$

$$= \frac{5p+2}{6} \cdot \frac{5}{3(5p+2)}$$

$$= \frac{5}{18}$$

$$= \frac{q^2+2q}{5+q} \cdot \frac{3q-6}{4-q^2}$$

$$= \frac{q(q+2)}{5+q} \cdot \frac{3(q-2)}{(2-q)(2+q)}$$

$$= \frac{q(q+2)3(q-2)}{(-1)(5+q)(q-2)(q+2)}$$

$$= -\frac{3q}{5+q}$$