7.4 Equations with Rational Expressions and Graphs

Objectives

1 Determine the domain of the variable in a rational equation.

2 Solve rational equations.

3 Recognize the graph of a rational function.

PEARSON

Determine the domain of the variable in a rational equation.

A **rational equation** is an equation that contains at least one rational expression with a variable in the denominator.

The **domain of the variable in a rational equation** is the intersection of the domains of the rational expressions in the equation.

CLASSROOM EXAMPLE 1 **Determining the Domains of the Variables in Rational Equations**

Find the domain of the variable in each equation.

Solution:

$$\frac{3}{x}+\frac{1}{4}=\frac{9}{4x}$$

The domains of the three rational expressions are, $\{x|x \neq 0\}$, $(-\infty, \infty)$, and $\{x|x \neq 0\}$. The intersection of these three domains is all real numbers except 0, which may be written $\{x|x \neq 0\}$.

$$\frac{2}{x^2-4}+\frac{1}{x+2}=\frac{1}{x-2}$$

The domains of the three rational expressions are, $\{x|x \neq \pm 2\}$, $\{x|x \neq 2\}$, $\{x|x \neq -2\}$. The domain is all real numbers except 2 and -2, $\{x|x \neq \pm 2\}$.

Objective 2

Solve rational equations.

Solve rational expressions.

To solve rational equations, we multiply all terms in the equation by the LCD to clear the fractions. ***We can do this only with equations, not expressions.***

Solving an Equation with Rational Expressions

Step 1 **Determine the domain of the variable.**

Step 2 **Multiply each side of the equation by the LCD** to clear the fractions.

Step 3 **Solve** the resulting equation.

Step 4 **Check** that each proposed solution is in the domain, and discard any values that are not. Check the remaining proposed solution(s) in the original equation.

CLASSROOM EXAMPLE 2 **Solving a Rational Equation**

Solve.

$$-\frac{3}{20}+\frac{2}{x}=\frac{5}{4x}$$

The domain, excludes 0.

Solution:

$$20x\left(-\frac{3}{20}+\frac{2}{x}\right)=20x\left(\frac{5}{4x}\right)$$ **Multiply by the LCD, 20*x*.**

$$20x\left(-\frac{3}{20}\right)+20x\left(\frac{2}{x}\right)=20x\left(\frac{5}{4x}\right)$$

$$-3x+40=25$$

$$-3x=-15$$

$$x=5$$ ← **Proposed solution**

CLASSROOM EXAMPLE 2

Solving a Rational Equation (cont'd)

Check.

$$-\frac{3}{20}+\frac{2}{x}=\frac{5}{4x}$$

$$-\frac{3}{20}+\frac{2}{5}=\frac{5}{4(5)}$$

$$-\frac{3}{20}+\frac{2}{5}=\frac{5}{20}$$

$$-\frac{3}{20}+\frac{8}{20}=\frac{5}{20}$$

$$\frac{5}{20}=\frac{5}{20}$$

The solution set is {5}.

CAUTION: When each side of an equation is multiplied by a ***variable*** expression, the resulting "solutions" may not satisfy the original equation. ***You must either determine and observe the domain or check all proposed solutions in the original equation. It is wise to do both.***

CLASSROOM EXAMPLE 3

Solving a Rational Equation with No Solution

Solve.

$$\frac{3}{x+1}=\frac{1}{x-1}-\frac{2}{x^2-1}$$

Solution:

The domain, excludes ±1. The LCD is $(x+1)(x-1)$.

$$(x+1)(x-1)\left(\frac{3}{x+1}\right)=(x+1)(x-1)\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$$

$$3(x-1)=x+1-2$$

$$3x-3=x-1$$

$$2x=2$$

$$x=1$$ ← **Proposed solution**

CLASSROOM EXAMPLE 3

Solving a Rational Equation with No Solution (cont'd)

Since the proposed solution is not in the domain, it cannot be an actual solution of the equation. Substituting 1 into the original equation shows why.

$$\frac{3}{x+1}=\frac{1}{x-1}-\frac{2}{x^2-1}$$

$$\frac{3}{1+1}=\frac{1}{1-1}-\frac{2}{1^2-1}$$

$$\frac{3}{2}=\frac{1}{0}-\frac{2}{0}$$

Division by 0 is undefined. The equation has no solution and the solution set is ∅.

CLASSROOM EXAMPLE 4

Solving a Rational Equation

Solve.

$$\frac{4}{x^2+x-6}-\frac{1}{x^2-4}=\frac{2}{x^2+5x+6}$$

Solution:

Factor each denominator.

$x^2 + x - 6 = (x + 3)(x - 2)$, so $x \neq -3, 2$.

$x^2 - 4 = (x + 2)(x - 2)$, so $x \neq \pm 2$.

$x^2 + 5x + 6 = (x + 3)(x + 2)$, so $x \neq -3, -2$.

The domain is $\{x \mid x \neq -3, \pm 2\}$.

The LCD = $(x + 3)(x + 2)(x - 2)$.

CLASSROOM EXAMPLE 4

Solving a Rational Equation (cont'd)

$$(x+3)(x+2)(x-2)\left[\frac{4}{x^2+x-6}-\frac{1}{x^2-4}\right]=(x+3)(x+2)(x-2)\frac{2}{x^2+5x+6}$$

$$4(x+2)-(x+3)=2(x-2)$$

$$4x+8-x-3=2x-4$$

$$3x+5=2x-4$$

$$x=-9$$ ← **Proposed solution**

The solution checks in the original equation. The solution set is {–9}.

CLASSROOM EXAMPLE 5

Solving a Rational Equation

Solve.

$$\frac{2}{x+3}-\frac{1}{x-1}=\frac{-x^2-3x}{x^2+2x-3}$$

Solution:

Factor each denominator to find the LCD.

$x + 3 = 0$, so $x \neq -3$.

$x - 1 = 0$, so $x \neq 1$.

$x^2 + 2x - 3 = (x + 3)(x - 1)$, so $x \neq -3, 1$.

The domain is $\{x \mid x \neq -3, 1\}$.

The LCD = $(x + 3)(x - 1)$.

CLASSROOM EXAMPLE 5 Solving a Rational Equation (cont'd)

$$(x+3)(x-1)\left(\frac{2}{x+3}-\frac{1}{x-1}\right)=(x+3)(x-1)\left(\frac{-x^2-3x}{x^2+2x-3}\right)$$

$$2(x-1)-1(x+3)=-x^2-3x$$
$$2x-2-x-3=-x^2-3x$$
$$x-5=-x^2-3x$$
$$x^2+4x-5=0$$
$$(x+5)(x-1)=0$$
$$x+5=0 \quad \text{or} \quad x-1=0$$
$$x=-5 \quad \text{or} \quad x=1$$

CLASSROOM EXAMPLE 5 Solving a Rational Equation (cont'd)

$$\frac{2}{x+3}-\frac{1}{x-1}=\frac{-x^2-3x}{x^2+2x-3}$$

$$x=-5 \quad \text{or} \quad x=1$$

Because 1 is not in the domain of the equation, it is not a solution.

$$\frac{2}{-5+3}-\frac{1}{-5-1}=\frac{-5^2-3(-5)}{(-5)^2+2(-5)-3}$$
$$\frac{2}{-2}-\frac{1}{-6}=\frac{-25+15}{25-10-3}$$
$$-1+\frac{1}{6}=-\frac{10}{12} \qquad -\frac{5}{6}=-\frac{5}{6}$$

The solution set is {–5}.

Objective 3

Recognize the graph of a rational function.

Recognize the graph of a rational function.

A function defined by a quotient of polynomials is a **rational function.** Because one or more values of *x* may be excluded from the domain of a rational function, their graphs are often **discontinuous.** That is, there will be one or more breaks in the graph.

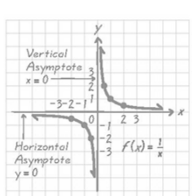
One simple rational function, defined by $f(x)=\frac{1}{x}$ and graphed on the next slide, is the **reciprocal function.** The domain of this function includes all real numbers except 0. Thus, this function pairs every real number except 0 with its reciprocal.

Since the domain of this function includes all real numbers except 0, there is no point on the graph with *x* = 0. The vertical line with equation *x* = 0 is called a **vertical asymptote** of the graph. Also, the horizontal line with equation *y* = 0 is called a **horizontal asymptote.**

Recognize the graph of a rational function.

The closer negative values of *x* are to 0, the smaller ("more negative") *y* is. → ← The closer positive values of *x* are to 0, the larger *y* is.

x	−3	−2	−1	−0.5	−0.25	−0.1	0.1	0.25	0.5	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	−1	−2	−4	−10	10	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$



Reciprocal function

$f(x)=\frac{1}{x}$

Domain: $\{x|x \neq 0\}$
Range: $\{x|x \neq 0\}$

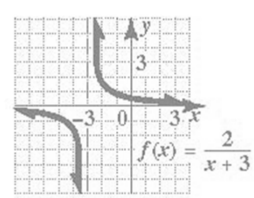
In general, if the *y*-values of a rational function approach ∞ or −∞ as the *x*-values approach a real number *a*, the vertical line *x* = *a* is a vertical asymptote of the graph. Also, if the *x*-values approach a real number *b* as $|x|$ increases without bound, the horizontal line *y* = *b* is a horizontal asymptote of the graph.

CLASSROOM EXAMPLE 6 Graphing a Rational Function

Graph, and give the equations of the vertical and horizontal asymptotes.

$$f(x)=\frac{2}{x+3}$$

Solution:



The vertical asymptote: *x* = − 3
The horizontal asymptote: *y* = 0