8.2 Rational Exponents

Objectives

1. Use exponential notation for *n*th roots.
2. Define and use expressions of the form $a^{m/n}$.
3. Convert between radicals and rational exponents.
4. Use the rules for exponents with rational exponents.

Use exponential notation for *n*th roots.

$a^{1/n}$

If $\sqrt[n]{a}$ is a real number, then $a^{1/n} = \sqrt[n]{a}$.

Notice that the denominator of the rational exponent is the index of the radical.

CLASSROOM EXAMPLE 1 **Evaluating Exponentials of the Form $a^{1/n}$**

Evaluate each exponential.

Solution:

$32^{1/5} = \sqrt[5]{32} = 2$

$64^{1/2} = \sqrt[2]{64} = \sqrt{64} = 8$

$-81^{1/4} = -\sqrt[4]{81} = -3$

$(-81)^{1/4} = \sqrt[4]{-81}$ Is not a real number because the radicand, –81, is negative and the index, 4, is even.

$(-64)^{1/3} = \sqrt[3]{-64} = -4$

$\left(\frac{1}{27}\right)^{1/3} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$

Define and use expressions of the form $a^{m/n}$.

$a^{m/n}$

If *m* and *n* are positive integers with *m/n* in lowest terms, then

$$a^{m/n} = \left(a^{1/n}\right)^m,$$

provided that $a^{1/n}$ is a real number. If $a^{1/n}$ is not a real number, then $a^{m/n}$ is not a real number.

CLASSROOM EXAMPLE 2 **Evaluating Exponentials of the Form $a^{m/n}$**

Evaluate each exponential.

Solution:

$25^{3/2} = \left(25^{1/2}\right)^3 = \left(\sqrt{25}\right)^3 = 5^3 = 125$

$27^{2/3} = \left(27^{1/3}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

$-16^{3/2} = -\left(16^{1/2}\right)^3 = -\left(\sqrt{16}\right)^3 = -4^3 = -64$

$(-64)^{2/3} = \left[(-64)^{1/3}\right]^2 = \left(\sqrt[3]{-64}\right)^2 = (-4)^2 = 16$

$(-36)^{3/2}$ is not a real number, since $(-36)^{1/2}$, or $\sqrt{-36}$, is not a real number.

Define and use expressions of the form $a^{m/n}$.

$a^{-m/n}$

If $a^{m/n}$ is a real number, then $a^{-m/n} = \frac{1}{a^{m/n}} \quad (a \neq 0)$.

CAUTION ***A negative exponent does not necessarily lead to a negative result. Negative exponents lead to reciprocals, which may be positive.***

CLASSROOM EXAMPLE 3 — **Evaluating Exponentials with Negative Rational Exponents**

Evaluate each exponential.

Solution:

$$81^{-3/4} = \frac{1}{81^{3/4}} = \frac{1}{\left(81^{1/4}\right)^3} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$$

$$36^{-3/2} = \frac{1}{36^{3/2}} = \frac{1}{\left(36^{1/2}\right)^3} = \frac{1}{\left(\sqrt{36}\right)^3} = \frac{1}{6^3} = \frac{1}{216}$$

$$\left(\frac{64}{25}\right)^{-3/2} = \left(\frac{25}{64}\right)^{3/2} = \left(\sqrt{\frac{25}{64}}\right)^3 = \left(\frac{5}{8}\right)^3 = \frac{125}{512}$$

Define and use expressions of the form $a^{m/n}$.

$a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}.$$

Define and use expressions of the form $a^{m/n}$.

Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

That is, raise *a* to the *m*th power and then take the *n*th root, or take the *n*th root of *a* and then raise to the *m*th power.

CLASSROOM EXAMPLE 4 — **Converting between Rational Exponents and Radicals**

Write each exponential as a radical. Assume that all variables, represent positive real numbers.

Solution:

$$19^{1/2} = \left(\sqrt[2]{19}\right)^1 = \sqrt{19}$$

$$11^{3/4} = \left(\sqrt[4]{11}\right)^3$$

$$14x^{2/3} = 14\left(\sqrt[3]{x}\right)^2$$

$$5x^{3/5} - (2x)^{3/5} = 5\left(\sqrt[5]{x}\right)^3 - \left(\sqrt[5]{2x}\right)^3$$

$$x^{-5/7} = \frac{1}{x^{5/7}} = \frac{1}{\left(\sqrt[7]{x}\right)^5}$$

$$\left(x^2 + y^2\right)^{1/3} = \sqrt[3]{x^2 + y^2}$$

CLASSROOM EXAMPLE 4 — **Converting between Rational Exponents and Radicals (cont'd)**

Write each radical as an exponential.

Solution:

$$\sqrt{37} = 37^{1/2}$$

$$\sqrt[4]{9^8} = 9^{8/4} = 9^2 = 81$$

$\sqrt[8]{z^8}$ = *z*, since *z* is assumed to be positive.

Use the rules for exponents with rational exponents.

Rules for Rational Exponents

Let *r* and *s* be rational numbers. For all real numbers *a* and *b* for which the indicated expressions exist,

$$a^r \cdot a^s = a^{r+s} \qquad a^{-r} = \frac{1}{a^r} \qquad \frac{a^r}{b^s} = a^{r-s}$$

$$\left(a^r\right)^s = a^{rs} \qquad (ab)^r = a^r b^r$$

$$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad \left(\frac{a}{b}\right)^{-r} = \frac{b^r}{a^r} \qquad a^{-r} = \left(\frac{1}{a}\right)^r$$

CLASSROOM EXAMPLE 5 — **Applying Rules for Rational Exponents**

Write with only positive exponents. Assume that all variables represent positive real numbers.

Solution:

$$3^{1/2} \cdot 3^{1/3} = 3^{1/2+1/3} = 3^{3/6+2/6} = 3^{5/6}$$

$$\frac{7^{2/3}}{7^{4/3}} = 7^{2/3-4/3} = 7^{-2/3} = \frac{1}{7^{2/3}}$$

$$\left(\frac{a^{1/3}b^{2/3}}{b}\right)^6 = \left(a^{1/3}b^{2/3-1}\right)^6 = \left(a^{1/3}b^{-1/3}\right)^6 = \left(a^{1/3}\right)^6\left(b^{-1/3}\right)^6$$

$$= a^{(1/3)6}b^{(-1/3)6} = a^{6/3}b^{-6/3} = a^2b^{-2} = \frac{a^2}{b^2}$$

CLASSROOM EXAMPLE 5 — **Applying Rules for Rational Exponents (cont'd)**

Write with only positive exponents. Assume that all variables represent positive real numbers.

Solution:

$$\left(\frac{a^3b^{-4}}{a^{-2}b^{1/5}}\right)^{-1/2} = \left(a^{3-(-2)}b^{-4-1/5}\right)^{-1/2} = \left(a^5b^{-21/5}\right)^{-1/2}$$

$$= \left(a^5\right)^{-1/2}\left(b^{-21/5}\right)^{-1/2} = a^{-5/2}b^{21/10} = \frac{b^{21/10}}{a^{5/2}}$$

$$r^{2/5}\left(r^{3/5}+r^{8/5}\right) = r^{2/5}\cdot r^{3/5} + r^{2/5}\cdot r^{8/5}$$

$$= r^{2/5+3/5} + r^{2/5+8/5} = r^{5/5} + r^{10/5} = r + r^2$$

CLASSROOM EXAMPLE 6 — **Applying Rules for Rational Exponents**

Write all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt[4]{x^3}\cdot\sqrt[5]{x} = x^{3/4}\cdot x^{1/5} = x^{3/4+1/5} = x^{15/20+4/20} = x^{19/20}$$

$$\frac{\sqrt{x^5}}{\sqrt[3]{x}} = \frac{x^{5/2}}{x^{1/3}} = x^{5/2-1/3} = x^{15/6-2/6} = x^{13/6}$$

$$\sqrt[3]{\sqrt[6]{x}} = \sqrt[3]{x^{1/6}} = \left(x^{1/6}\right)^{1/3} = x^{(1/6)(1/3)} = x^{1/18}$$