8.5 Multiplying and Dividing Radical Expressions

Objectives

1 Multiply radical expressions.

2 Rationalize denominators with one radical term.

3 Rationalize denominators with binomials involving radicals.

4 Write radical quotients in lowest terms.

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Multiply radical expressions.

We multiply binomial expressions involving radicals by using the **FOIL** method from **Section 5.4.** Recall that this method refers to multiplying the **F**irst terms, **O**uter terms, **I**nner terms, and **L**ast terms of the binomials.

CLASSROOM EXAMPLE 1 **Multiplying Binomials Involving Radical Expressions**

Multiply, using the FOIL method.

Solution:

F O I L

$$\left(2+\sqrt{3}\right)\left(1+\sqrt{5}\right) = 2 + 2\sqrt{5} + 1\sqrt{3} + \sqrt{15}$$

$$\left(4+\sqrt{5}\right)\left(4-\sqrt{5}\right) = 16 - 4\sqrt{5} + 4\sqrt{5} - 5 = 11$$

This is a difference of squares.

$$\left(\sqrt{13}-2\right)^2 = \left(\sqrt{13}-2\right)\left(\sqrt{13}-2\right)$$

$$= 13 - 2\sqrt{13} - 2\sqrt{13} + 4$$

$$= 17 - 4\sqrt{13}$$

CLASSROOM EXAMPLE 1 **Multiplying Binomials Involving Radical Expressions (cont'd)**

Multiply, using the FOIL method.

Solution:

$$\left(4+\sqrt[3]{7}\right)\left(4-\sqrt[3]{7}\right) = 16 - 4\sqrt[3]{7} + 4\sqrt[3]{7} - \sqrt[3]{7^2}$$

$$= 16 - \sqrt[3]{49}$$

$$\left(\sqrt{r}+\sqrt{s}\right)\left(\sqrt{r}-\sqrt{s}\right) = \left(\sqrt{r}\right)^2 - \left(\sqrt{s}\right)^2$$

$r \geq 0$ and $s \geq 0$

$$= r - s$$

Difference of squares

Rationalize denominators with one radical.

Rationalizing the Denominator

A common way of "standardizing" the form of a radical expression is to have the denominator contain no radicals. The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalizing the denominator**. This is done by multiplying y a form of 1.

CLASSROOM EXAMPLE 2 **Rationalizing Denominators with Square Roots**

Rationalize each denominator.

Solution:

$$\frac{5}{\sqrt{11}}$$

Multiply the numerator and denominator by the denominator. This is in effect multiplying by 1.

$$= \frac{5 \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}} = \frac{5\sqrt{11}}{11}$$

$$\frac{5\sqrt{6}}{\sqrt{5}} = \frac{5\sqrt{6} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{5\sqrt{30}}{5} = \sqrt{30}$$

CLASSROOM EXAMPLE 2 — **Rationalizing Denominators with Square Roots (cont'd)**

Rationalize the denominator.

Solution:

$$\frac{-8}{\sqrt{18}} = \frac{-8\cdot\sqrt{18}}{\sqrt{18}\cdot\sqrt{18}} = \frac{-8\sqrt{18}}{18} = \frac{-8\sqrt{9\cdot 2}}{18} = \frac{-24\sqrt{2}}{18} = \frac{-4\sqrt{2}}{3}$$

CLASSROOM EXAMPLE 3 — **Rationalizing Denominators in Roots of Fractions**

Simplify the radical.

Solution:

$$-\sqrt{\frac{8}{45}} = -\frac{\sqrt{8}}{\sqrt{45}}$$

$$= -\frac{\sqrt{4\cdot 2}}{\sqrt{9\cdot 5}}$$ **Factor.**

$$= -\frac{2\sqrt{2}}{3\sqrt{5}}$$ **Product Rule**

$$= -\frac{2\sqrt{2}\cdot\sqrt{5}}{3\sqrt{5}\cdot\sqrt{5}}$$ **Multiply by radical in denominator.**

$$= -\frac{2\sqrt{10}}{3\cdot 5}$$ **Product Rule**

$$= -\frac{2\sqrt{10}}{15}$$

CLASSROOM EXAMPLE 3 — **Rationalizing Denominators in Roots of Fractions (cont'd)**

Simplify the radical.

Solution:

$$\sqrt{\frac{200k^6}{y^7}}, y > 0 = \frac{\sqrt{200k^6}}{\sqrt{y^7}} = -\frac{\sqrt{100\cdot 2\cdot (k^3)^2}}{\sqrt{y^6\cdot y}} = \frac{10k^3\sqrt{2}}{y^3\sqrt{y}}$$

$$= \frac{10k^3\sqrt{2}\cdot\sqrt{y}}{y^3\sqrt{y}\cdot\sqrt{y}} = \frac{10k^3\sqrt{2y}}{y^4}$$

CLASSROOM EXAMPLE 4 — **Rationalizing Denominators with Cube and Fourth Roots**

Simplify.

Solution:

$$\sqrt[3]{\frac{15}{32}} = \frac{\sqrt[3]{15}}{\sqrt[3]{32}} = \frac{\sqrt[3]{15}}{\sqrt[3]{8}\cdot\sqrt[3]{4}} = \frac{\sqrt[3]{15}}{2\sqrt[3]{4}} = \frac{\sqrt[3]{15}\cdot\sqrt[3]{2}}{2\sqrt[3]{4}\cdot\sqrt[3]{2}} = \frac{\sqrt[3]{30}}{2\cdot\sqrt[3]{8}} = \frac{\sqrt[3]{30}}{4}$$

$$\sqrt[4]{\frac{6y}{w^2}}, y \ge 0, w > 0 = \frac{\sqrt[4]{6y}}{\sqrt[4]{w^2}} = \frac{\sqrt[4]{6y}\cdot\sqrt[4]{w^2}}{\sqrt[4]{w^2}\cdot\sqrt[4]{w^2}} = \frac{\sqrt[4]{6yw^2}}{w}$$

Rationalize denominators with binomials involving radicals.

To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

$$\frac{3}{1+\sqrt{2}}$$

we must use ***conjugates.*** The conjugate of $1+\sqrt{2}$ is $1-\sqrt{2}$.

In general, x + y and x − y are **conjugates.**

Rationalize denominators with binomials involving radicals.

Rationalizing a Binomial Denominator

Whenever a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.

CLASSROOM EXAMPLE 5 — Rationalizing Binomial Denominators

Rationalize the denominator.

Solution:

$$\frac{4}{2-\sqrt{3}} = \frac{4\left(2+\sqrt{3}\right)}{2-\sqrt{3}\left(2+\sqrt{3}\right)}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{4-3}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{1}$$

$$= 4(2+\sqrt{3}), \text{ or } 8+4\sqrt{3}$$

CLASSROOM EXAMPLE 5 — Rationalizing Binomial Denominators (cont'd)

Rationalize the denominator.

Solution:

$$\frac{7}{\sqrt{2}+\sqrt{13}} = \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{\left(\sqrt{2}+\sqrt{13}\right)\left(\sqrt{2}-\sqrt{13}\right)}$$

$$= \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{2-13}$$

$$= \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{-11}$$

$$= \frac{-7\left(\sqrt{2}-\sqrt{13}\right)}{11}$$

CLASSROOM EXAMPLE 5 — Rationalizing Binomial Denominators (cont'd)

Rationalize the denominator.

Solution:

$$\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{7}} = \frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{7}} \cdot \frac{\sqrt{2}+\sqrt{7}}{\sqrt{2}+\sqrt{7}}$$

$$= \frac{\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}}{2-7}$$

$$= \frac{\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}}{-5}$$

$$= \frac{-\left(\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}\right)}{5}$$

CLASSROOM EXAMPLE 5 — Rationalizing Binomial Denominators (cont'd)

Rationalize the denominator.

Solution:

$$\frac{2}{\sqrt{k}+\sqrt{z}}, \quad k \neq z, k>0, z>0$$

$$= \frac{2\left(\sqrt{k}-\sqrt{z}\right)}{\left(\sqrt{k}+\sqrt{z}\right)\left(\sqrt{k}-\sqrt{z}\right)}$$

$$= \frac{2\left(\sqrt{k}-\sqrt{z}\right)}{k-z}$$

CLASSROOM EXAMPLE 6 — Writing Radical Quotients in Lowest Terms

Write the quotient in lowest terms.

Solution:

$$\frac{24-36\sqrt{7}}{16} = \frac{12\left(2-3\sqrt{7}\right)}{16}$$ **Factor the numerator and denominator.**

$$= \frac{4\cdot 3\left(2-3\sqrt{7}\right)}{4\cdot 4}$$ **Divide out common factors.**

$$= \frac{3\left(2-3\sqrt{7}\right)}{4} \text{ or } = \frac{6-9\sqrt{7}}{4}$$

CAUTION *Be careful to factor before writing a quotient in lowest terms.*

CLASSROOM EXAMPLE 6 — Writing Radical Quotients in Lowest Terms (cont'd)

Write the quotient in lowest terms.

Solution:

$$\frac{2x+\sqrt{32x^2}}{6x}, x>0 = \frac{2x+4x\sqrt{2}}{6x}$$ **Product rule**

$$= \frac{2x\left(1+2\sqrt{2}\right)}{6x}$$ **Factor the numerator.**

$$= \frac{2x\left(1+2\sqrt{2}\right)}{2\cdot 3x}$$ **Divide out common factors.**

$$= \frac{1+2\sqrt{2}}{3}$$