8.1 Radical Expressions and Graphs

Objectives

1. Find roots of numbers.
2. Find principal roots.
3. Graph functions defined by radical expressions.
4. Find *n*th roots of *n*th powers.
5. Use a calculator to find roots.

PEARSON

Find roots of numbers.

The opposite (or inverse) of ***squaring*** a number is taking its ***square root.***

$$\sqrt{36} = 6, \text{ because } 6^2 = 36.$$

We now extend our discussion of roots to include ***cube roots*** $\sqrt[3]{\ }$, ***fourth roots*** $\sqrt[4]{\ }$, and higher roots.

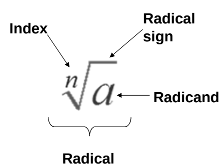
$$\sqrt[n]{a}$$

The *n*th root of *a*, written $\sqrt[n]{a}$, is a number whose *n*th power equals *a*. That is,

$$\sqrt[n]{a} = b \text{ means } b^n = a.$$

Find roots of numbers.

The number *a* is the **radicand.**
n is the **index** or **order**.
The expression is the **radical.**



CLASSROOM EXAMPLE 1 **Simplifying Higher Roots**

Simplify.
Solution:

$$\sqrt[3]{27} = 3, \text{ because } 3^3 = 27$$

$$\sqrt[3]{216} = 6, \text{ because } 6^3 = 216$$

$$\sqrt[4]{256} = 4, \text{ because } 4^4 = 256$$

$$\sqrt[5]{243} = 3, \text{ because } 3^5 = 243$$

$$\sqrt[4]{\frac{16}{81}} = \frac{2}{3}, \text{ because } \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$\sqrt[3]{0.064} = 0.4, \text{ because } 0.4^3 = 0.064$$

Find principal roots.

***n*th Root**

Case 1 If *n* is ***even*** and *a* is ***positive or 0,*** then

$\sqrt[n]{a}$ represents the **principal *n*th root** of *a*,

$-\sqrt[n]{a}$ represents the **negative *n*th root** of *a*.

Case 2 If *n* is ***even*** and *a* is ***negative,*** then

$\sqrt[n]{a}$ is not a real number.

Case 3 If *n* is ***odd***, then

there is exactly one real *n*th root of *a*, written $\sqrt[n]{a}$.

CLASSROOM EXAMPLE 2 **Finding Roots**

Find each root.

Solution:

$\sqrt{36} = 6$ $\qquad$ $-\sqrt{36} = -6$

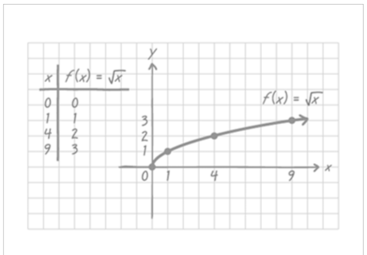
$\sqrt[4]{16} = 2$ $\qquad$ $-\sqrt[4]{16} = -2$

$\sqrt[4]{-16}$ Not a real number. $\qquad$ $\sqrt[5]{243} = 3$

$\sqrt[5]{-243} = -3$

Graph functions defined by radical expressions.

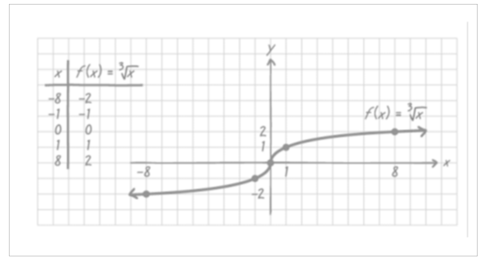
Square Root Function



The domain and range of the square root function are [0, ∞).

Graph functions defined by radical expressions.

Cube Root Function



The domain and range of the cube function are (–∞, ∞).

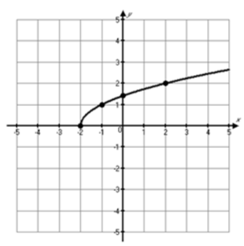
CLASSROOM EXAMPLE 3 **Graphing Functions Defined with Radicals**

Graph the function by creating a table of values. Give the domain and range.

$f(x)=\sqrt{x+2}$

Solution:

x	*f*(*x*)
–2	$\sqrt{-2+2}=0$
–1	$\sqrt{-1+2}=1$
0	$\sqrt{0+2}=1.41$
2	$\sqrt{2+2}=2$



Domain: [–2, ∞)

Range: [0, ∞)

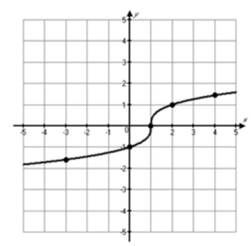
CLASSROOM EXAMPLE 3 **Graphing Functions Defined with Radicals (cont'd)**

Graph the function by creating a table of values. Give the domain and range.

$f(x)=\sqrt[3]{x-1}$

Solution:

X	*f*(*x*)
0	$\sqrt[3]{0-1}=-1$
1	$\sqrt[3]{1-1}=0$
2	$\sqrt[3]{2-1}=1$
–3	$\sqrt[3]{-3-1}=-1.587$
4	$\sqrt[3]{4-1}=1.44$



Domain: (–∞, ∞)

Range: (–∞, ∞)

Objective 4

Find *n*th roots of *n*th powers.

Find *n*th roots of *n*th powers.

$\sqrt{a^2}$

For any real number *a*, $\sqrt{a^2}=|a|$.

That is, the principal square root of a^2 is the absolute value of *a*.

CLASSROOM EXAMPLE 4

Simplifying Square Roots by Using Absolute Value

Find each square root.

Solution:

$\sqrt{15^2} = |15| = 15$ $\quad$ $\sqrt{(-12)^2} = |-12| = 12$

$\sqrt{y^2} = |y|$ $\quad$ $\sqrt{(-y^2)} = |-y| = |y|$

Find *n*th roots of *n*th powers.

$$\sqrt[n]{a^n}$$

If *n* is an ***even*** positive integer, then $\sqrt[n]{a^n} = |a|$.

If *n* is an ***odd*** positive integer, then $\sqrt[n]{a^n} = a.$

That is, use absolute value when *n* is even; absolute value is not necessary when *n* is odd.

CLASSROOM EXAMPLE 5

Simplifying Higher Roots by Using Absolute Value

Simplify each root.

Solution:

$\sqrt[4]{(-5)^4} = |-5| = 5$

$\sqrt[5]{(-5)^5} = -5 \quad n \text{ is odd}$

$-\sqrt[6]{(-3)^6} = -|-3| = -3$

$-\sqrt[4]{m^8} = -m^2 \quad n \text{ is even}$

$\sqrt[3]{x^{24}} = x^8$

$\sqrt[6]{y^{18}} = \sqrt[6]{(y^3)^6} = |y^3|$

Objective 5

Use a calculator to find roots.

CLASSROOM EXAMPLE 6

Finding Approximations for Roots

Use a calculator to approximate each radical to three decimal places.

Solution:

$\sqrt{17} = 4.123$ $\quad$ $-\sqrt{362} = -19.026$

$\sqrt[3]{9482} = 21.166$ $\quad$ $\sqrt[4]{6825} = 9.089$

CLASSROOM EXAMPLE 7

Using Roots to Calculate Resonant Frequency

In electronics, the resonant frequency *f* of a circuit may be found by the formula $f = \frac{1}{2\pi\sqrt{LC}}$ where *f* is the cycles per second, *L* is in henrys, and *C* is in farads. (Henrys and farads are units of measure in electronics). Find the resonant frequency *f* if $L = 6 \times 10^{-5}$ and $C = 4 \times 10^{-9}$.

Solution:

$$f = \frac{1}{2\pi\sqrt{LC}} \quad f = \frac{1}{2\pi\sqrt{(6\times10^{-5})(4\times10^{-9})}} \approx 324,874$$

About 325,000 cycles per second.

8.2 Rational Exponents

Objectives

1 Use exponential notation for *n*th roots.

2 Define and use expressions of the form $a^{m/n}$.

3 Convert between radicals and rational exponents.

4 Use the rules for exponents with rational exponents.

PEARSON

Use exponential notation for *n*th roots.

$a^{1/n}$

If $\sqrt[n]{a}$ is a real number, then $a^{1/n} = \sqrt[n]{a}$.

Notice that the denominator of the rational exponent is the index of the radical.

CLASSROOM EXAMPLE 1 **Evaluating Exponentials of the Form $a^{1/n}$**

Evaluate each exponential.

Solution:

$32^{1/5} = \sqrt[5]{32} = 2$

$64^{1/2} = \sqrt[2]{64} = \sqrt{64} = 8$

$-81^{1/4} = -\sqrt[4]{81} = -3$

$(-81)^{1/4} = \sqrt[4]{-81}$ Is not a real number because the radicand, –81, is negative and the index, 4, is even.

$(-64)^{1/3} = \sqrt[3]{-64} = -4$

$\left(\frac{1}{27}\right)^{1/3} = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$

Define and use expressions of the form $a^{m/n}$.

$a^{m/n}$

If *m* and *n* are positive integers with *m/n* in lowest terms, then

$$a^{m/n} = \left(a^{1/n}\right)^m,$$

provided that $a^{1/n}$ is a real number. If $a^{1/n}$ is not a real number, then $a^{m/n}$ is not a real number.

CLASSROOM EXAMPLE 2 **Evaluating Exponentials of the Form $a^{m/n}$**

Evaluate each exponential.

Solution:

$25^{3/2} = \left(25^{1/2}\right)^3 = \left(\sqrt{25}\right)^3 = 5^3 = 125$

$27^{2/3} = \left(27^{1/3}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

$-16^{3/2} = -\left(16^{1/2}\right)^3 = -\left(\sqrt{16}\right)^3 = -4^3 = -64$

$(-64)^{2/3} = \left[(-64)^{1/3}\right]^2 = \left(\sqrt[3]{-64}\right)^2 = (-4)^2 = 16$

$(-36)^{3/2}$ is not a real number, since $(-36)^{1/2}$, or $\sqrt{-36}$, is not a real number.

Define and use expressions of the form $a^{m/n}$.

$a^{-m/n}$

If $a^{m/n}$ is a real number, then $a^{-m/n} = \frac{1}{a^{m/n}} \quad (a \neq 0)$.

CAUTION ***A negative exponent does not necessarily lead to a negative result. Negative exponents lead to reciprocals, which may be positive.***

CLASSROOM EXAMPLE 3 — Evaluating Exponentials with Negative Rational Exponents

Evaluate each exponential.

Solution:

$$81^{-3/4} = \frac{1}{81^{3/4}} = \frac{1}{\left(81^{1/4}\right)^3} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$$

$$36^{-3/2} = \frac{1}{36^{3/2}} = \frac{1}{\left(36^{1/2}\right)^3} = \frac{1}{\left(\sqrt{36}\right)^3} = \frac{1}{6^3} = \frac{1}{216}$$

$$\left(\frac{64}{25}\right)^{-3/2} = \left(\frac{25}{64}\right)^{3/2} = \left(\sqrt{\frac{25}{64}}\right)^3 = \left(\frac{5}{8}\right)^3 = \frac{125}{512}$$

Define and use expressions of the form $a^{m/n}$.

$a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}.$$

Define and use expressions of the form $a^{m/n}$.

Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

That is, raise *a* to the *m*th power and then take the *n*th root, or take the *n*th root of *a* and then raise to the *m*th power.

CLASSROOM EXAMPLE 4 — Converting between Rational Exponents and Radicals

Write each exponential as a radical. Assume that all variables, represent positive real numbers.

Solution:

$$19^{1/2} = \left(\sqrt[2]{19}\right)^1 = \sqrt{19}$$

$$11^{3/4} = \left(\sqrt[4]{11}\right)^3$$

$$14x^{2/3} = 14\left(\sqrt[3]{x}\right)^2$$

$$5x^{3/5} - (2x)^{3/5} = 5\left(\sqrt[5]{x}\right)^3 - \left(\sqrt[5]{2x}\right)^3$$

$$x^{-5/7} = \frac{1}{x^{5/7}} = \frac{1}{\left(\sqrt[7]{x}\right)^5}$$

$$\left(x^2 + y^2\right)^{1/3} = \sqrt[3]{x^2 + y^2}$$

CLASSROOM EXAMPLE 4 — Converting between Rational Exponents and Radicals (cont'd)

Write each radical as an exponential.

Solution:

$$\sqrt{37} = 37^{1/2}$$

$$\sqrt[4]{9^8} = 9^{8/4} = 9^2 = 81$$

$\sqrt[8]{z^8}$ = *z*, since *z* is assumed to be positive.

Use the rules for exponents with rational exponents.

Rules for Rational Exponents

Let *r* and *s* be rational numbers. For all real numbers *a* and *b* for which the indicated expressions exist,

$$a^r \cdot a^s = a^{r+s} \qquad a^{-r} = \frac{1}{a^r} \qquad \frac{a^r}{b^s} = a^{r-s}$$

$$\left(a^r\right)^s = a^{rs} \qquad (ab)^r = a^r b^r$$

$$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad \left(\frac{a}{b}\right)^{-r} = \frac{b^r}{a^r} \qquad a^{-r} = \left(\frac{1}{a}\right)^r$$

CLASSROOM EXAMPLE 5 **Applying Rules for Rational Exponents**

Write with only positive exponents. Assume that all variables represent positive real numbers.

Solution:

$$3^{1/2}\cdot 3^{1/3} = 3^{1/2+1/3} = 3^{3/6+2/6} = 3^{5/6}$$

$$\frac{7^{2/3}}{7^{4/3}} = 7^{2/3-4/3} = 7^{-2/3} = \frac{1}{7^{2/3}}$$

$$\left(\frac{a^{1/3}b^{2/3}}{b}\right)^6 = \left(a^{1/3}b^{2/3-1}\right)^6 = \left(a^{1/3}b^{-1/3}\right)^6 = \left(a^{1/3}\right)^6\left(b^{-1/3}\right)^6$$

$$= a^{(1/3)6}b^{(-1/3)6} = a^{6/3}b^{-6/3} = a^2b^{-2} = \frac{a^2}{b^2}$$

CLASSROOM EXAMPLE 5 **Applying Rules for Rational Exponents (cont'd)**

Write with only positive exponents. Assume that all variables represent positive real numbers.

Solution:

$$\left(\frac{a^3b^{-4}}{a^{-2}b^{1/5}}\right)^{-1/2} = \left(a^{3-(-2)}b^{-4-1/5}\right)^{-1/2} = \left(a^5b^{-21/5}\right)^{-1/2}$$

$$= \left(a^5\right)^{-1/2}\left(b^{-21/5}\right)^{-1/2} = a^{-5/2}b^{21/10} = \frac{b^{21/10}}{a^{5/2}}$$

$$r^{-2/5}\left(r^{3/5}+r^{8/5}\right) = r^{2/5}\cdot r^{3/5} + r^{2/5}\cdot r^{8/5}$$

$$= r^{2/5+3/5} + r^{2/5+8/5} = r^{5/5} + r^{10/5} = r + r^2$$

CLASSROOM EXAMPLE 6 **Applying Rules for Rational Exponents**

Write all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt[4]{x^3}\cdot\sqrt[5]{x} = x^{3/4}\cdot x^{1/5} = x^{3/4+1/5} = x^{15/20+4/20} = x^{19/20}$$

$$\frac{\sqrt{x^5}}{\sqrt[3]{x}} = \frac{x^{5/2}}{x^{1/3}} = x^{5/2-1/3} = x^{15/6-2/6} = x^{13/6}$$

$$\sqrt[3]{\sqrt[6]{x}} = \sqrt[3]{x^{1/6}} = \left(x^{1/6}\right)^{1/3} = x^{(1/6)(1/3)} = x^{1/18}$$

8.3 Simplifying Radical Expressions

Objectives

1 Use the product rule for radicals.

2 Use the quotient rule for radicals.

3 Simplify radicals.

4 Simplify products and quotients of radicals with different indexes.

5 Use the Pythagorean theorem.

6 Use the distance formula.

PEARSON

Use the product rule for radicals.

Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

That is, the product of two *n*th roots is the *n*th root of the product.

CAUTION ***Use the product rule only when the radicals have the same index.***

CLASSROOM EXAMPLE 1 **Using the Product Rule**

Multiply. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt{5} \cdot \sqrt{13} = \sqrt{5 \cdot 13} = \sqrt{65}$$

$$\sqrt{7} \cdot \sqrt{xy} = \sqrt{7xy}$$

CLASSROOM EXAMPLE 2 **Using the Product Rule**

Multiply. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt[3]{2} \cdot \sqrt[3]{7} = \sqrt[3]{2 \cdot 7} = \sqrt[3]{14}$$

$$\sqrt[6]{8r^2} \cdot \sqrt[6]{2r^3} = \sqrt[6]{16r^5}$$

$$\sqrt[5]{9y^2x} \cdot \sqrt[5]{8xy^2} = \sqrt[5]{72y^4x^2}$$

$\sqrt{7} \cdot \sqrt[3]{5}$ This expression cannot be simplified by using the product rule.

Use the quotient rule for radicals.

Quotient Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number,

then $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

That is, the *n*th root of a quotient is the quotient of the *n*th roots.

CLASSROOM EXAMPLE 3 **Using the Quotient Rule**

Simplify. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt{\frac{100}{81}} = \frac{10}{9}$$

$$\sqrt{\frac{11}{25}} = \frac{\sqrt{11}}{5}$$

$$\sqrt[3]{\frac{18}{125}} = \frac{\sqrt[3]{18}}{\sqrt[3]{125}} = \frac{\sqrt[3]{18}}{5}$$

$$\sqrt{\frac{y^8}{16}} = \frac{\sqrt{y^8}}{\sqrt{16}} = \frac{y^4}{4}$$

$$-\sqrt[3]{\frac{x^2}{r^{12}}} = -\frac{\sqrt[3]{x^2}}{\sqrt[3]{r^{12}}} = -\frac{\sqrt[3]{x^2}}{r^4}$$

CLASSROOM EXAMPLE 4 — **Simplifying Roots of Numbers**

Simplify.

Solution:

$\sqrt{32} = \sqrt{16\cdot 2} = \sqrt{16}\cdot\sqrt{2} = 4\sqrt{2}$

$\sqrt{300} = \sqrt{100\cdot 3} = \sqrt{100}\cdot\sqrt{3} = 10\sqrt{3}$

$\sqrt{35}$ Cannot be simplified further.

$\sqrt[3]{54} = \sqrt[3]{27\cdot 2} = \sqrt[3]{27}\cdot\sqrt[3]{2} = 3\sqrt[3]{2}$

$\sqrt[4]{243} = \sqrt[4]{3^4\cdot 3} = 3\sqrt[4]{3}$

Simplify radicals.

Conditions for a Simplified Radical

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.

CAUTION ***Be careful with which factors belong outside the radical sign and which belong inside.***

CLASSROOM EXAMPLE 5 — **Simplifying Radicals Involving Variables**

Simplify. Assume that all variables represent positive real numbers.

Solution:

$\sqrt{25p^7} = \sqrt{5^2\cdot(p^3)^2\cdot p} = 5p^3\sqrt{p}$

$\sqrt{72y^3x} = \sqrt{36\cdot 2\cdot y^2\cdot y\cdot x} = 6y\sqrt{2yx}$

$\sqrt[3]{-27y^7x^5z^6} = \sqrt[3]{-3^3\cdot y^6\cdot y\cdot x^3\cdot x^2\cdot z^6} = -3y^2xz^2\sqrt[3]{yx^2}$

$-\sqrt[4]{32a^5b^7} = -\sqrt[4]{2^4\cdot 2\cdot a^4\cdot a\cdot b^4\cdot b^3} = -2ab\sqrt[4]{2ab^3}$

CLASSROOM EXAMPLE 6 — **Simplifying Radicals by Using Smaller Indexes**

Simplify. Assume that all variables represent positive real numbers.

Solution:

$\sqrt[12]{2^3} = \left(2^3\right)^{1/12} = 2^{1/4} = \sqrt[4]{2}$

$\sqrt[6]{t^2} = \left(t^2\right)^{1/6} = \left(t\right)^{1/3} = \sqrt[3]{t}$

Simplify radicals.

$\sqrt[kn]{a^{km}}$

If m is an integer, n and k are natural numbers, and all indicated roots exist, then

$$\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}.$$

CLASSROOM EXAMPLE 7 — **Multiplying Radicals with Different Indexes**

Simplify.

Solution:

$\sqrt{5}\cdot\sqrt[3]{4}$

The indexes, 2 and 3, have a least common index of 6, use rational exponents to write each radical as a sixth root.

$= 5^{1/2} = 5^{3/6} = \sqrt[6]{5^3} = \sqrt[6]{125}$

$= 4^{1/3} = 4^{2/6} = \sqrt[6]{4^2} = \sqrt[6]{16}$

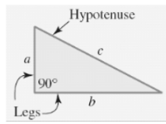
$\sqrt{5}\cdot\sqrt[3]{4} = \sqrt[6]{125}\cdot\sqrt[6]{16} = \sqrt[6]{2000}$

Use the Pythagorean theorem.

Pythagorean Theorem

If c is the length of the longest side of a right triangle and a and b are lengths of the shorter sides, then

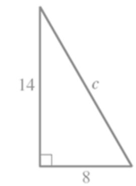
$c^2 = a^2 + b^2$.



The longest side is the **hypotenuse**, and the two shorter sides are the **legs**, of the triangle. The hypotenuse is the side opposite the right angle.

CLASSROOM EXAMPLE 8 **Using the Pythagorean Theorem**

Find the length of the unknown side of the triangle.



Solution:

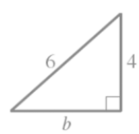
$c^2 = a^2 + b^2$
$c^2 = 14^2 + 8^2$
$c^2 = 196 + 64$
$c^2 = 260$

$$c = \sqrt{260}$$
$$c = \sqrt{4 \cdot 65}$$
$$c = \sqrt{4} \cdot \sqrt{65}$$
$$c = 2\sqrt{65}$$

The length of the hypotenuse is $2\sqrt{65}$.

CLASSROOM EXAMPLE 8 **Using the Pythagorean Theorem (cont'd)**

Find the length of the unknown side of the triangle.



Solution:

$c^2 = a^2 + b^2$
$6^2 = 4^2 + b^2$
$36 = 16 + b^2$
$20 = b^2$

$$\sqrt{20} = b$$
$$\sqrt{4 \cdot 5} = b$$
$$\sqrt{4} \cdot \sqrt{5} = b$$
$$2\sqrt{5} = b$$

The length of the leg is $2\sqrt{5}$.

Use the distance formula.

Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

CLASSROOM EXAMPLE 9 **Using the Distance Formula**

Find the distance between each pair of points.
(2, –1) and (5, 3)
Solution:
Designate which points are (x_1, y_1) and (x_2, y_2).
$(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (5, 3)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(5-2)^2 + (3-(-1))^2}$$
$$d = \sqrt{(3)^2 + (4)^2}$$
$$d = \sqrt{9+16}$$
$$d = \sqrt{25} = 5$$

Start with the *x*-value and *y*-value of the *same* point.

CLASSROOM EXAMPLE 9 **Using the Distance Formula (cont'd)**

Find the distance between each pairs of points.
(–3, 2) and (0, –4)
Solution:
Designate which points are (x_1, y_1) and (x_2, y_2).
$(x_1, y_1) = (-3, 2)$ and $(x_2, y_2) = (0, -4)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$$
$$d = \sqrt{(0-(-3))^2 + (-4-2)^2}$$
$$d = \sqrt{(3)^2 + (-6)^2}$$
$$d = \sqrt{9+36}$$
$$d = \sqrt{45} = 3\sqrt{5}$$

8.4 Adding and Subtracting Radical Expressions

Objective

1 Simplify radical expressions involving addition and subtraction.

PEARSON

Simplify radical expressions involving addition and subtraction.

CAUTION ***Only radical expressions with the same index and the same radicand may be combined.*** Expressions such as $3\sqrt{3}+2\sqrt[3]{3}$ or $5\sqrt{3}+2\sqrt{2}$ cannot be simplified by combining terms.

CLASSROOM EXAMPLE 1 **Adding and Subtracting Radicals**

Add or subtract to simplify each radical expression.

Solution:

$3\sqrt{5}+7\sqrt{5} = (3+7)\sqrt{5} = 10\sqrt{5}$

$$2\sqrt{11}-\sqrt{11}+3\sqrt{44} = 2\sqrt{11}-\sqrt{11}+3\sqrt{4}\cdot\sqrt{11}$$
$$= 2\sqrt{11}-1\sqrt{11}+3\cdot 2\cdot\sqrt{11}$$
$$= (2-1+6)\sqrt{11}$$
$$= 7\sqrt{11}$$

CLASSROOM EXAMPLE 1 **Adding and Subtracting Radicals (cont'd)**

Add or subtract to simplify each radical expression.

Solution:

$$5\sqrt{12y}+6\sqrt{75y},\ y\ge 0 = 5\sqrt{4}\cdot\sqrt{3y}+6\sqrt{25}\cdot\sqrt{3y}$$
$$= 5\cdot 2\sqrt{3y}+6\cdot 5\sqrt{3y}$$
$$= 10\sqrt{3y}+30\sqrt{3y}$$
$$= (10+30)\sqrt{3y}$$
$$= 40\sqrt{3y}$$

$9\sqrt{5}-4\sqrt{10}$ This expression can not be simplified any further.

Simplify radical expressions involving addition and subtraction.

CAUTION Do not confuse the product rule with combining like terms. ***The root of a sum does not equal the sum of the roots.*** For example $\sqrt{9+16}\ne\sqrt{9}+\sqrt{16}$

since $\sqrt{9+16}=\sqrt{25}=5,$ but $\sqrt{9}+\sqrt{16}=3+4=7.$

CLASSROOM EXAMPLE 2 **Adding and Subtracting Radicals with Higher Indexes**

Add or subtract to simplify the radical expression. Assume that all variables represent positive real numbers.

Solution:

$$-2\sqrt[4]{32}-7\sqrt[4]{162} = -2\sqrt[4]{16\cdot 2}-7\sqrt[4]{81\cdot 2}$$
$$= -2\cdot 2\sqrt[4]{2}-7\cdot 3\sqrt[4]{2}$$
$$= -4\sqrt[4]{2}-21\sqrt[4]{2}$$
$$= -25\sqrt[4]{2}$$

CLASSROOM EXAMPLE 2 — **Adding and Subtracting Radicals with Higher Indexes (cont'd)**

Add or subtract to simplify the radical expression. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt[3]{p^4q^7} - \sqrt[3]{64pq} = \sqrt[3]{p^3q^6 \cdot pq} - \sqrt[3]{64 \cdot pq}$$

$$= pq^2\sqrt[3]{pq} - 4\sqrt[3]{pq}$$

$$= \left(pq^2 - 4\right)\sqrt[3]{pq}$$

CLASSROOM EXAMPLE 2 — **Adding and Subtracting Radicals with Higher Indexes (cont'd)**

Add or subtract to simplify the radical expression. Assume that all variables represent positive real numbers.

Solution:

$$6\sqrt[3]{16z^7} + 4\sqrt{200z^5} = 6\sqrt[3]{8z^6 \cdot 2z} + 4\sqrt{100z^4 \cdot 2z}$$

$$= 6\sqrt[3]{8z^6} \cdot \sqrt[3]{2z} + 4\sqrt{100z^4} \cdot \sqrt{2z}$$

$$= 6 \cdot 2z^2\sqrt[3]{2z} + 4 \cdot 10z^2\sqrt{2z}$$

$$= 12z^2\sqrt[3]{2z} + 40z^2\sqrt{2z}$$

CLASSROOM EXAMPLE 3 — **Adding and Subtracting Radicals with Fractions**

Perform the indicated operations. Assume that all variables represent positive real numbers.

Solution:

$$2\sqrt{\frac{32}{36}} + 2\frac{\sqrt{27}}{\sqrt{108}} = 2\frac{\sqrt{16 \cdot 2}}{\sqrt{36}} + 2\frac{\sqrt{9 \cdot 3}}{\sqrt{36 \cdot 3}}$$

$$= 2\left(\frac{4\sqrt{2}}{6}\right) + 2\left(\frac{3\sqrt{3}}{6\sqrt{3}}\right)$$

$$= \frac{4\sqrt{2}}{3} + \frac{3}{3} \qquad = \frac{4\sqrt{2} + 3}{3}$$

CLASSROOM EXAMPLE 3 — **Adding and Subtracting Radicals with Fractions (cont'd)**

Perform the indicated operations. Assume that all variables represent positive real numbers.

Solution:

$$\sqrt{\frac{80}{y^4}} + \sqrt{\frac{81}{y^{10}}} = \frac{\sqrt{16 \cdot 5}}{\sqrt{y^4}} + \frac{\sqrt{81}}{\sqrt{y^{10}}}$$

$$= \frac{4\sqrt{5}}{y^2} + \frac{9}{y^5}$$

$$= \frac{4y^3\sqrt{5}}{y^5} + \frac{9}{y^5} \qquad = \frac{4y^3\sqrt{5} + 9}{y^5}$$

8.5 Multiplying and Dividing Radical Expressions

Objectives

1. Multiply radical expressions.
2. Rationalize denominators with one radical term.
3. Rationalize denominators with binomials involving radicals.
4. Write radical quotients in lowest terms.

PEARSON

Multiply radical expressions.

We multiply binomial expressions involving radicals by using the **FOIL** method from **Section 5.4.** Recall that this method refers to multiplying the **F**irst terms, **O**uter terms, **I**nner terms, and **L**ast terms of the binomials.

CLASSROOM EXAMPLE 1 **Multiplying Binomials Involving Radical Expressions**

Multiply, using the FOIL method.

Solution:

F O I L

$$\left(2+\sqrt{3}\right)\left(1+\sqrt{5}\right) = 2 + 2\sqrt{5} + 1\sqrt{3} + \sqrt{15}$$

$$\left(4+\sqrt{5}\right)\left(4-\sqrt{5}\right) = 16 - 4\sqrt{5} + 4\sqrt{5} - 5 = 11$$

This is a difference of squares.

$$\left(\sqrt{13}-2\right)^2 = \left(\sqrt{13}-2\right)\left(\sqrt{13}-2\right)$$

$$= 13 - 2\sqrt{13} - 2\sqrt{13} + 4$$

$$= 17 - 4\sqrt{13}$$

CLASSROOM EXAMPLE 1 **Multiplying Binomials Involving Radical Expressions (cont'd)**

Multiply, using the FOIL method.

Solution:

$$\left(4+\sqrt[3]{7}\right)\left(4-\sqrt[3]{7}\right) = 16 - 4\sqrt[3]{7} + 4\sqrt[3]{7} - \sqrt[3]{7^2}$$

$$= 16 - \sqrt[3]{49}$$

$$\left(\sqrt{r}+\sqrt{s}\right)\left(\sqrt{r}-\sqrt{s}\right) = \left(\sqrt{r}\right)^2 - \left(\sqrt{s}\right)^2$$

$r \ge 0$ and $s \ge 0$

$$= r - s$$

Difference of squares

Rationalize denominators with one radical.

Rationalizing the Denominator

A common way of "standardizing" the form of a radical expression is to have the denominator contain no radicals. The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalizing the denominator**. This is done by multiplying y a form of 1.

CLASSROOM EXAMPLE 2 **Rationalizing Denominators with Square Roots**

Rationalize each denominator.

Solution:

$$\frac{5}{\sqrt{11}}$$

Multiply the numerator and denominator by the denominator. This is in effect multiplying by 1.

$$= \frac{5\cdot\sqrt{11}}{\sqrt{11}\cdot\sqrt{11}} = \frac{5\sqrt{11}}{11}$$

$$\frac{5\sqrt{6}}{\sqrt{5}} = \frac{5\sqrt{6}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{5\sqrt{30}}{5} = \sqrt{30}$$

CLASSROOM EXAMPLE 2 — **Rationalizing Denominators with Square Roots (cont'd)**

Rationalize the denominator.

Solution:

$$\frac{-8}{\sqrt{18}} = \frac{-8\cdot\sqrt{18}}{\sqrt{18}\cdot\sqrt{18}} = \frac{-8\sqrt{18}}{18} = \frac{-8\sqrt{9\cdot 2}}{18} = \frac{-24\sqrt{2}}{18} = \frac{-4\sqrt{2}}{3}$$

CLASSROOM EXAMPLE 3 — **Rationalizing Denominators in Roots of Fractions**

Simplify the radical.

Solution:

$$-\sqrt{\frac{8}{45}} = -\frac{\sqrt{8}}{\sqrt{45}}$$

$$= -\frac{\sqrt{4\cdot 2}}{\sqrt{9\cdot 5}}$$ **Factor.**

$$= -\frac{2\sqrt{2}}{3\sqrt{5}}$$ **Product Rule**

$$= -\frac{2\sqrt{2}\cdot\sqrt{5}}{3\sqrt{5}\cdot\sqrt{5}}$$ **Multiply by radical in denominator.**

$$= -\frac{2\sqrt{10}}{3\cdot 5}$$ **Product Rule**

$$= -\frac{2\sqrt{10}}{15}$$

CLASSROOM EXAMPLE 3 — **Rationalizing Denominators in Roots of Fractions (cont'd)**

Simplify the radical.

Solution:

$$\sqrt{\frac{200k^6}{y^7}}, y>0 \quad = \frac{\sqrt{200k^6}}{\sqrt{y^7}} = -\frac{\sqrt{100\cdot 2\cdot (k^3)^2}}{\sqrt{y^6\cdot y}} = \frac{10k^3\sqrt{2}}{y^3\sqrt{y}}$$

$$= \frac{10k^3\sqrt{2}\cdot\sqrt{y}}{y^3\sqrt{y}\cdot\sqrt{y}} = \frac{10k^3\sqrt{2y}}{y^4}$$

CLASSROOM EXAMPLE 4 — **Rationalizing Denominators with Cube and Fourth Roots**

Simplify. $\sqrt[3]{\frac{15}{32}}$

Solution:

$$= \frac{\sqrt[3]{15}}{\sqrt[3]{32}} = \frac{\sqrt[3]{15}}{\sqrt[3]{8}\cdot\sqrt[3]{4}} = \frac{\sqrt[3]{15}}{2\sqrt[3]{4}} = \frac{\sqrt[3]{15}\cdot\sqrt[3]{2}}{2\sqrt[3]{4}\cdot\sqrt[3]{2}} = \frac{\sqrt[3]{30}}{2\cdot\sqrt[3]{8}} = \frac{\sqrt[3]{30}}{4}$$

$$\sqrt[4]{\frac{6y}{w^2}}, y\ge 0, w>0$$

$$= \frac{\sqrt[4]{6y}}{\sqrt[4]{w^2}} = \frac{\sqrt[4]{6y}\cdot\sqrt[4]{w^2}}{\sqrt[4]{w^2}\cdot\sqrt[4]{w^2}} = \frac{\sqrt[4]{6yw^2}}{w}$$

Rationalize denominators with binomials involving radicals.

To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

$$\frac{3}{1+\sqrt{2}}$$

we must use ***conjugates.*** The conjugate of $1+\sqrt{2}$ is $1-\sqrt{2}$.

In general, x + y and x − y are **conjugates.**

Rationalize denominators with binomials involving radicals.

Rationalizing a Binomial Denominator

Whenever a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.

CLASSROOM EXAMPLE 5 — **Rationalizing Binomial Denominators**

Rationalize the denominator.

Solution:

$$\frac{4}{2-\sqrt{3}} = \frac{4\left(2+\sqrt{3}\right)}{2-\sqrt{3}\left(2+\sqrt{3}\right)}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{4-3}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{1}$$

$$= 4(2+\sqrt{3}), \text{ or } 8+4\sqrt{3}$$

CLASSROOM EXAMPLE 5 — **Rationalizing Binomial Denominators (cont'd)**

Rationalize the denominator.

Solution:

$$\frac{7}{\sqrt{2}+\sqrt{13}} = \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{\left(\sqrt{2}+\sqrt{13}\right)\left(\sqrt{2}-\sqrt{13}\right)}$$

$$= \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{2-13}$$

$$= \frac{7\left(\sqrt{2}-\sqrt{13}\right)}{-11}$$

$$= \frac{-7\left(\sqrt{2}-\sqrt{13}\right)}{11}$$

CLASSROOM EXAMPLE 5 — **Rationalizing Binomial Denominators (cont'd)**

Rationalize the denominator.

Solution:

$$\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{7}} = \frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{7}} \cdot \frac{\sqrt{2}+\sqrt{7}}{\sqrt{2}+\sqrt{7}}$$

$$= \frac{\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}}{2-7}$$

$$= \frac{\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}}{-5}$$

$$= \frac{-\left(\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35}\right)}{5}$$

CLASSROOM EXAMPLE 5 — **Rationalizing Binomial Denominators (cont'd)**

Rationalize the denominator.

Solution:

$$\frac{2}{\sqrt{k}+\sqrt{z}}, \quad k \neq z, k > 0, z > 0$$

$$= \frac{2\left(\sqrt{k}-\sqrt{z}\right)}{\left(\sqrt{k}+\sqrt{z}\right)\left(\sqrt{k}-\sqrt{z}\right)}$$

$$= \frac{2\left(\sqrt{k}-\sqrt{z}\right)}{k-z}$$

CLASSROOM EXAMPLE 6 — **Writing Radical Quotients in Lowest Terms**

Write the quotient in lowest terms.

Solution:

$$\frac{24-36\sqrt{7}}{16} = \frac{12\left(2-3\sqrt{7}\right)}{16}$$ **Factor the numerator and denominator.**

$$= \frac{4\cdot 3\left(2-3\sqrt{7}\right)}{4\cdot 4}$$ **Divide out common factors.**

$$= \frac{3\left(2-3\sqrt{7}\right)}{4} \text{ or } = \frac{6-9\sqrt{7}}{4}$$

CAUTION ***Be careful to factor before writing a quotient in lowest terms.***

CLASSROOM EXAMPLE 6 — **Writing Radical Quotients in Lowest Terms (cont'd)**

Write the quotient in lowest terms.

Solution:

$$\frac{2x+\sqrt{32x^2}}{6x}, x > 0 = \frac{2x+4x\sqrt{2}}{6x}$$ **Product rule**

$$= \frac{2x\left(1+2\sqrt{2}\right)}{6x}$$ **Factor the numerator.**

$$= \frac{2x\left(1+2\sqrt{2}\right)}{2\cdot 3x}$$ **Divide out common factors.**

$$= \frac{1+2\sqrt{2}}{3}$$

8.6 Solving Equations with Radicals

Objectives

1. Solve radical equations by using the power rule.
2. Solve radical equations that require additional steps.
3. Solve radical equations with indexes greater than 2.
4. Use the power rule to solve a formula for a specified variable.

PEARSON

Solve radical equations by using the power rule.

Power Rule for Solving an Equation with Radicals

If both sides of an equation are raised to the same power, all solutions of the original equation are also solutions of the new equation.

The power rule does not say that all solutions of the new equation are solutions of the original equation. They may or may not be. Solutions that do not satisfy the ***original*** equation are called **extraneous solutions.** They must be rejected.

CAUTION When the power rule is used to solve an equation, ***every solution of the new equation must be checked in the original equation***.

CLASSROOM EXAMPLE 1 **Using the Power Rule**

Solve $\sqrt{5x+1}=4.$

Solution:

$$\left(\sqrt{5x+1}\right)^2=4^2$$
$$5x+1=16$$
$$5x=15$$
$$x=3$$

Check:

$$\sqrt{5x+1}=4$$
$$\sqrt{5\cdot 3+1}=4$$
$$\sqrt{16}=4$$
$$4=4$$

True

Since 3 satisfies the ***original*** equation, the solution set is {3}.

Solve radical equations by using the power rule.

Solving an Equation with Radicals

Step 1 **Isolate the radical**. Make sure that one radical term is alone on one side of the equation.

Step 2 **Apply the power rule**. Raise each side of the equation to a power that is the same as the index of the radical.

Step 3 **Solve** the resulting equation; if it still contains a radical, repeat ***Steps 1 and 2.***

Step 4 **Check** all proposed solutions in the original equation.

CLASSROOM EXAMPLE 2 **Using the Power Rule**

Solve $\sqrt{5x+3}+2=0.$

Solution:

$$\sqrt{5x+3}=-2$$

The equation has no solution, because the square root of a real number must be nonnegative.

The solution set is ∅.

Objective 2

Solve radical equations that require additional steps.

CLASSROOM EXAMPLE 3 **Using the Power Rule (Squaring a Binomial)**

Solve $\sqrt{5-x}=x+1.$

Solution:

Step 1 The radical is alone on the left side of the equation.

Step 2 Square both sides. $\left(\sqrt{5-x}\right)^2=\left(x+1\right)^2$

$$5-x=x^2+2x+1$$

Step 3 The new equation is quadratic, so get 0 on one side.

$$0=x^2+3x-4$$
$$0=(x+4)(x-1)$$
$$x+4=0 \quad \text{or} \quad x-1=0$$
$$x=-4 \quad \text{or} \quad x=1$$

CLASSROOM EXAMPLE 3 **Using the Power Rule (Squaring a Binomial) (cont'd)**

Step 4 Check each proposed solution in the original equation.

$x=-4$	or $x=1$
$\sqrt{5-x}=x+1$	$\sqrt{5-x}=x+1$
$\sqrt{5-(-4)}=(-4)+1$	$\sqrt{5-(1)}=(1)+1$
$\sqrt{9}=-3$	$\sqrt{4}=2$
$3\neq-3$	$2=2$
False	**True**

The solution set is {1}. The other proposed solution, –4, is extraneous.

CLASSROOM EXAMPLE 4 **Using the Power Rule (Squaring a Binomial)**

Solve $\sqrt{1-2x-x^2}=x+1.$

Solution:

Step 1 The radical is alone on the left side of the equation.

Step 2 Square both sides. $\left(\sqrt{1-2x-x^2}\right)^2=\left(x+1\right)^2$

$$1-2x-x^2=x^2+2x+1$$

Step 3 The new equation is quadratic, so get 0 on one side.

$$0=2x^2+4x$$
$$0=2x(x+2)$$
$$2x=0 \quad \text{or} \quad x+2=0$$
$$x=0 \quad \text{or} \quad x=-2$$

CLASSROOM EXAMPLE 4 **Using the Power Rule (Squaring a Binomial) (cont'd)**

Step 4 Check each proposed solution in the original equation.

$x=0$	or $x=-2$
$\sqrt{1-2x-x^2}=x+1$	$\sqrt{1-2x-x^2}=x+1$
$\sqrt{1-2(0)-0^2}=0+1$	$\sqrt{1-2(-2)-(-2)^2}=-2+1$
$\sqrt{1}=1$	$\sqrt{1}\neq-1$
$1=1$	
True	**False**

The solution set of the original equation is {0}.

CLASSROOM EXAMPLE 5 **Using the Power Rule (Squaring Twice)**

Solve $\sqrt{2x+3}+\sqrt{x+1}=1.$

Solution:

$$\sqrt{2x+3}=1-\sqrt{x+1}$$

$\left(\sqrt{2x+3}\right)^2=\left(1-\sqrt{x+1}\right)^2$ **Square both sides.**

$$2x+3=1-2\sqrt{x+1}+(x+1)$$

$x+1=-2\sqrt{x+1}$ **Isolate the remaining radical.**

$\left(x+1\right)^2=\left(-2\sqrt{x+1}\right)^2$ **Square both sides.**

$x^2+2x+1=(-2)^2\left(\sqrt{x+1}\right)^2$ **Apply the exponent rule $(ab)^2=a^2b^2$.**

$$x^2+2x+1=4(x+1)$$

CLASSROOM EXAMPLE 5 **Using the Power Rule (Squaring Twice) (cont'd)**

$$x^2+2x+1=4(x+1)$$
$$x^2+2x+1=4x+4$$
$$x^2-2x-3=0$$
$$(x-3)(x+1)=0$$
$$x-3=0 \quad \text{or} \quad x+1=0$$
$$x=3 \quad \text{or} \quad x=-1$$

Check: x = 3

$$\sqrt{2(3)+3}+\sqrt{3+1}=1$$
$$\sqrt{6+3}+\sqrt{4}=1$$
$$\sqrt{9}+\sqrt{4}=1$$
$$3+2=1$$
$$5\neq1$$

False

Check: x = – 1

$$\sqrt{2(-1)+3}+\sqrt{-1+1}=1$$
$$\sqrt{1}+\sqrt{0}=1$$

True $1=1$

The solution set is {–1}.

Objective 3

Solve radical equations with indexes greater than 2.

CLASSROOM EXAMPLE 6 **Using the Power Rule for a Power Greater Than 2**

Solve $\sqrt[3]{2x+7} = \sqrt[3]{3x-2}.$

Solution:

$$\left(\sqrt[3]{2x+7}\right)^3 = \left(\sqrt[3]{3x-2}\right)^3$$ **Cube both sides.**

$$2x+7 = 3x-2$$

$$9 = x$$

Check:

$$\sqrt[3]{2(9)+7} = \sqrt[3]{3(9)-2}$$

$$\sqrt[3]{18+7} = \sqrt[3]{27-2}$$

$$\sqrt[3]{25} = \sqrt[3]{25}$$ **True**

The solution set is {9}.

Objective 4

Use the power rule to solve a formula for a specified variable.

CLASSROOM EXAMPLE 7 **Solving a Formula from Electronics for a Variable**

Solve the formula for *R.*

Solution:

$$Z = \sqrt{\frac{R}{T}}$$

$$(Z)^2 = \left(\sqrt{\frac{R}{T}}\right)^2$$

$$Z^2 = \frac{R}{T}$$

$$TZ^2 = R \text{ or } R = TZ^2$$

8.7 Complex Numbers

Objectives

1. Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.
2. Recognize subsets of the complex numbers.
3. Add and subtract complex numbers.
4. Multiply complex numbers.
5. Divide complex numbers.
6. Find powers of *i*.

PEARSON

Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.

Imaginary Unit *i*

The **imaginary unit *i*** is defined as

$$i = \sqrt{-1}, \quad \text{where} \quad i^2 = -1.$$

That is, *i* is the principal square root of –1.

Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.

$$\sqrt{-b}$$

For any positive real number *b*, $\sqrt{-b} = i\sqrt{b}$.

CAUTION: It is easy to mistake $\sqrt{2}i$ for $\sqrt{2i}$ with the *i* under the radical. For this reason, we usually write $\sqrt{2}i$ as $i\sqrt{2}$, as in the definition of $\sqrt{-b}$.

CLASSROOM EXAMPLE 1 — Simplifying Square Roots of Negative Numbers

Write each number as a product of a real number and *i*.

Solution:

$$\sqrt{-25} = i\sqrt{25} = 5i$$

$$-\sqrt{-81} = -i\sqrt{81} = -9i$$

$$\sqrt{-7} = i\sqrt{7}$$

$$\sqrt{-44} = i\sqrt{44} = i\sqrt{4\cdot 11} = 2i\sqrt{11}$$

CLASSROOM EXAMPLE 2 — Multiplying Square Roots of Negative Numbers

Multiply.

Solution:

$$\sqrt{-16}\cdot\sqrt{-25} = i\sqrt{16}\cdot i\sqrt{25} = i\cdot 4\cdot i\cdot 5 = 20i^2 = 20(-1) = -20$$

$$\sqrt{-8}\cdot\sqrt{-6} = i\sqrt{8}\cdot i\sqrt{6} = i^2\sqrt{8\cdot 6} = i^2\sqrt{48} = i^2\sqrt{16\cdot 3} = -4\sqrt{3}$$

$$\sqrt{-6}\cdot\sqrt{-5} = i\sqrt{6}\cdot i\sqrt{5} = i^2\sqrt{6\cdot 5} = (-1)\sqrt{30} = -\sqrt{30}$$

$$\sqrt{-5}\cdot\sqrt{7} = i\sqrt{5}\cdot\sqrt{7} = i\sqrt{35}$$

CLASSROOM EXAMPLE 3 — Dividing Square Roots of Negative Numbers

Divide.

Solution:

$$\frac{\sqrt{-80}}{\sqrt{-5}} = \frac{i\sqrt{80}}{i\sqrt{5}} = \sqrt{\frac{80}{5}} = \sqrt{16} = 4$$

$$\frac{\sqrt{-40}}{\sqrt{10}} = \frac{i\sqrt{40}}{\sqrt{10}} = i\sqrt{\frac{40}{10}} = i\sqrt{4} = 2i$$

Recognize subsets of the complex numbers.

For a complex number $a + bi$, if $b = 0$, then $a + bi = a$, which is a real number. In which case ***a*** is ***real part*** and ***b*** is ***imaginary part***.

Thus, the set of real numbers is a subset of the set of complex numbers.

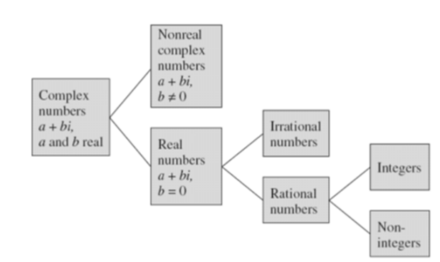
If $a = 0$ and $b \neq 0$, the complex number is said to be a **pure imaginary number**.

For example, $3i$ is a pure imaginary number. A number such as $7 + 2i$ is a **nonreal complex number**.

A complex number written in the form $a + bi$ is in **standard form**.

Recognize subsets of the complex numbers.

The relationships among the various sets of numbers.



CLASSROOM EXAMPLE 4 **Adding Complex Numbers**

Add.

Solution:

$$(-1-8i)+(9-3i) = (-1+9)+(-8-3)i$$
$$= 8-11i$$

$$(-3+2i)+(1-3i)+(-7-5i)$$
$$= [-3+1+(-7)]+[2+(-3)+(-5)]i$$
$$= -9-6i$$

CLASSROOM EXAMPLE 5 **Subtracting Complex Numbers**

Subtract.

Solution:

$$(-1+2i)-(4+i) \quad = (-1-4)+(2-1)i \quad = -5+i$$

$$(8-5i)-(12-3i) = (8-12)+[-5-(-3)]i$$
$$= (8-12)+(-5+3)i$$
$$= -4-2i$$

$$(-10+6i)-(-10+10i) = [-10-(-10)]+(6-10)i$$
$$= 0-4i \quad = -4i$$

CLASSROOM EXAMPLE 6 **Multiplying Complex Numbers**

Multiply.

Solution:

$$6i(4+3i) \quad = 6i(4)+6i(3i)$$
$$= 24i+18i^2$$
$$= 24i+18(-1)$$
$$= -18+24i$$

CLASSROOM EXAMPLE 6 **Multiplying Complex Numbers (cont'd)**

Multiply.

Solution:

$$(6-4i)(2+4i) = \underbrace{6(2)}_{First}+\underbrace{6(4i)}_{Outer}+\underbrace{(-4i)(2)}_{Inner}+\underbrace{(-4i)(4i)}_{Last}$$
$$= 12+24i-8i-16i^2$$
$$= 12+16i-16(-1)$$
$$= 12+16i+16$$
$$= 28+16i$$

CLASSROOM EXAMPLE 6 **Multiplying Complex Numbers (cont'd)**

Multiply.

Solution:

$$(3+2i)(3+4i) = \underbrace{3(3)}_{First} + \underbrace{3(4i)}_{Outer} + \underbrace{(2i)(3)}_{Inner} + \underbrace{(2i)(4i)}_{Last}$$

$$= 9 + 12i + 6i + 8i^2$$

$$= 9 + 18i + 8(-1)$$

$$= 9 + 18i - 8$$

$$= 1 + 18i$$

Multiply complex numbers.

The product of a complex number and its conjugate is always a real number.

$$(a + bi)(a - bi) = a^2 - b^2(-1)$$
$$= a^2 + b^2$$

CLASSROOM EXAMPLE 7 **Dividing Complex Numbers**

Find the quotient.

Solution:

$$\frac{23-i}{3-i} = \frac{(23-i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{69+23i-3i+1}{3^2+1}$$

$$= \frac{70+20i}{10}$$

$$= \frac{10(7+2i)}{10} \qquad = 7+2i$$

CLASSROOM EXAMPLE 7 **Dividing Complex Numbers (cont'd)**

Find the quotient.

Solution:

$$\frac{5-i}{i} = \frac{(5-i)(-i)}{i(-i)}$$

$$= \frac{-5i+i^2}{-i^2}$$

$$= \frac{-5i+(-1)}{-(-1)}$$

$$= \frac{-5i-1}{1} \qquad = -1-5i$$

Find powers of *i*.

Because $i^2 = -1$, we can find greater powers of *i*, as shown below.

$i^3 = i \cdot i^2 = i \cdot (-1) = -i$

$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$

$i^5 = i \cdot i^4 = i \cdot 1 = i$

$i^6 = i^2 \cdot i^4 = (-1) \cdot (1) = -1$

$i^7 = i^3 \cdot i^4 = (-i) \cdot (1) = -I$

$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

CLASSROOM EXAMPLE 8 **Simplifying Powers of *i***

Find each power of *i*.

Solution:

$$i^{28} = \left(i^4\right)^7 = 1^7 = 1$$

$$i^{19} = i^{16} \cdot i^3 = \left(i^4\right)^4 \cdot i^3 = 1^4 \cdot (-i) = -i$$

$$i^{-9} = \frac{1}{i^9} = \frac{1}{i^8 \cdot i} = \frac{1}{\left(i^4\right)^2 \cdot i} = \frac{1}{1^2 \cdot i} = \frac{1}{i}$$

$$= \frac{1(-i)}{i \cdot (-i)} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = \frac{-i}{1} = -i$$

$$i^{-22} = \frac{1}{i^{22}} = \frac{1}{i^{20} \cdot i^2} = \frac{1}{\left(i^4\right)^5 \cdot (-1)} = \frac{1}{1^5 \cdot (-1)} = \frac{1}{-1} = -1$$