9.1 The Square Root Property and Completing the Square

Objectives

1 Review the zero-factor property.

2 Learn the square root property.

3 Solve quadratic equations of the form $(ax + b)^2 = c$ by extending the square root property.

4 Solve quadratic equations by completing the square.

5 Solve quadratic equations with solutions that are not real numbers.

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The Square Root Property and Completing the Square

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where *a, b,* and *c* are real numbers, with $a \neq 0$, is a **quadratic equation**. The given form is called **standard form**.

Review the zero-factor property.

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

CLASSROOM EXAMPLE 1 **Using the Zero-Factor Property**

Use the zero-factor property to solve $2x^2 - 3x = -1$.

Solution:

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$2x = 1 \qquad\qquad x = 1$$

$$x = \frac{1}{2}$$

Solution set is $\left\{\frac{1}{2}, 1\right\}$.

Learn the square root property.

Square Root Property

If *x* and *k* are complex numbers and $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

CAUTION Remember that if $k \neq 0$, using the square root property always produces ***two*** square roots, one positive and one negative.

CLASSROOM EXAMPLE 2 **Using the Zero-Factor Property**

Solve each equation.

Solution:

$$m^2 = 64$$

By the square root property, $m = 8$ or $m = –8$.

The solution set is {–8, 8}.

$$3x^2 - 54 = 0$$

$$3x^2 = 54$$

$$x^2 = 18$$

By the square root property,

$$x = \sqrt{18} \quad \text{or} \quad x = -\sqrt{18},$$

$$x = 3\sqrt{2} \quad \text{or} \quad x = -3\sqrt{2}.$$

Check:

$$x = \pm 3\sqrt{2}: \quad 3(18) - 54 = 0$$

True

Solution set is $\left\{3\sqrt{2}, -3\sqrt{2}\right\}$.

CLASSROOM EXAMPLE 3 — **Using the Square Root Property in an Application**

An expert marksman can hold a silver dollar at forehead level, drop it, draw his gun, and shoot the coin as it passes waist level. If the coin falls about 4 ft, use the formula $d = 16t^2$ to find the time that elapses between the dropping of the coin and the shot.

Solution:

$$d = 16t^2$$

$$4 = 16t^2$$

$$\frac{1}{4} = t^2$$

By the square root property,

$$t = \frac{1}{2} \quad \text{or} \quad t = -\frac{1}{2}$$

Since time cannot be negative, we discard the negative solution. Therefore, 0.5 sec elapses between the dropping of the coin and the shot.

CLASSROOM EXAMPLE 4 — **Extending the Square Root Property**

Solve $(x - 3)^2 = 16$.

Solution:

$$x - 3 = \sqrt{16} \quad \text{or} \quad x - 3 = -\sqrt{16}$$

$$x - 3 = 4 \qquad\qquad x - 3 = -4$$

$$x = 7 \qquad\qquad x = -1$$

CLASSROOM EXAMPLE 4 — **Extending the Square Root Property (cont'd)**

Check:

$$(7-3)^2 = 16 \qquad\qquad (-1-3)^2 = 16$$

$$4^2 = 16 \qquad\qquad -4^2 = 16$$

$$16 = 16 \qquad\qquad 16 = 16$$

True | **True**

The solution set is $\{-1, 7\}$.

CLASSROOM EXAMPLE 5 — **Extending the Square Root Property**

Solve $(3x + 1)^2 = 2$.

Solution:

$$3x + 1 = \sqrt{2} \quad \text{or} \quad 3x + 1 = -\sqrt{2}$$

$$3x = -1 + \sqrt{2} \qquad\qquad 3x = -1 - \sqrt{2}$$

$$x = \frac{-1 + \sqrt{2}}{3} \qquad\qquad x = \frac{-1 - \sqrt{2}}{3}$$

CLASSROOM EXAMPLE 5 — **Extending the Square Root Property (cont'd)**

We show a check for the first solution. The check for the other solution is similar.

Check:

$$(3x + 1)^2 = 2$$

$$\left[3\left(\frac{-1+\sqrt{2}}{3}\right)+1\right]^2 = 2 \quad ? \quad \textbf{Let } x = \frac{-1+\sqrt{2}}{3}.$$

$$\left(-1+\sqrt{2}+1\right)^2 = 2 \quad ? \quad \textbf{Multiply.}$$

$$\left(\sqrt{2}\right)^2 = 2 \quad ? \quad \textbf{Simplify.}$$

$$2 = 2 \quad \textbf{True}$$

The solution set is $\left\{\frac{-1 \pm \sqrt{2}}{3}\right\}$.

CLASSROOM EXAMPLE 6 — **Solving a Quadratic Equation by Completing the Square ($a = 1$)**

Solve $x^2 - 2x - 10 = 0$.

Solution:

$$x^2 - 2x = 10$$

Completing the square

$$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1.$$

Add 1 to each side.

$$x^2 - 2x + 1 = 10 + 1$$

$$(x - 1)^2 = 11$$

CLASSROOM EXAMPLE 6 Solving a Quadratic Equation by Completing the Square (a = 1) (cont'd)

Use the square root property.

$$x-1=\sqrt{11} \quad \text{or} \quad x-1=-\sqrt{11}$$
$$x=1+\sqrt{11} \quad \text{or} \quad x=1-\sqrt{11}$$

Check: $x=1+\sqrt{11}$:

$$\left(1+\sqrt{11}\right)^2-2\left(1+\sqrt{11}\right)-10=0 \quad ?$$
$$12+2\sqrt{11}-2-2\sqrt{11}-10=0 \quad \textbf{True}$$

The solution set is $\left\{1\pm\sqrt{11}\right\}$.

Solve quadratic equations by completing the square.

Completing the Square

To solve $ax^2 + bx + c = 0$ ($a \neq 0$) by completing the square, use these steps.

Step 1 **Be sure the second-degree (squared) term has coefficient 1.** If the coefficient of the squared term is one, proceed to ***Step 2.*** If the coefficient of the squared term is not 1 but some other nonzero number a, divide each side of the equation by a.

Step 2 **Write the equation in correct form** so that terms with variables are on one side of the equals symbol and the constant is on the other side.

Step 3 **Square half the coefficient of the first-degree (linear) term.**

Solve quadratic equations by completing the square.

Completing the Square (continued)

Step 4 **Add the square to each side.**

Step 5 **Factor the perfect square trinomial.** One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

Step 6 **Solve the equation.** Apply the square root property to complete the solution.

Steps 1 and 2 can be done in either order. With some equations, it is more convenient to do ***Step 2*** first.

CLASSROOM EXAMPLE 7 Solving a Quadratic Equation by Completing the Square (a = 1)

Solve $x^2 + 3x - 1 = 0$.

Solution:

$$x^2 + 3x = 1$$

Completing the square.

$$\left[\frac{1}{2}(3)\right]^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

Add the square to each side.

$$x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4}$$

$$\left(x+\frac{3}{2}\right)^2=\frac{13}{4}$$

CLASSROOM EXAMPLE 7 Solving a Quadratic Equation by Completing the Square (a = 1) (cont'd)

Use the square root property.

$$x+\frac{3}{2}=\sqrt{\frac{13}{4}} \quad \text{or} \quad x+\frac{3}{2}=-\sqrt{\frac{13}{4}}$$
$$x+\frac{3}{2}=\frac{\sqrt{13}}{2} \quad \text{or} \quad x+\frac{3}{2}=-\frac{\sqrt{13}}{2}$$
$$x=\frac{-3+\sqrt{13}}{2} \quad \text{or} \quad x=\frac{-3-\sqrt{13}}{2}$$

Check that the solution set is $\left\{\frac{-3\pm\sqrt{13}}{2}\right\}$.

CLASSROOM EXAMPLE 8 Solving a Quadratic Equation by Completing the Square ($a \neq 1$)

Solve $3x^2 + 6x - 2 = 0$.

Solution:

$$3x^2 + 6x = 2$$
$$x^2+2x=\frac{2}{3}$$

Completing the square.

$$\left[\tfrac{1}{2}(2)\right]^2=(1)^2=1$$

$$x^2+2x+1=\frac{2}{3}+1$$

$$(x+1)^2=\frac{5}{3}$$

CLASSROOM EXAMPLE 8 — **Solving a Quadratic Equation by Completing the Square ($a \neq 1$) (cont'd)**

Use the square root property.

$$x+1=\sqrt{\frac{5}{3}} \quad \text{or} \quad x+1\frac{3}{2}=-\sqrt{\frac{5}{3}}$$

$$x=-1+\sqrt{\frac{5}{3}} \quad \text{or} \quad x=-1-\sqrt{\frac{5}{3}}$$

$$x=-1+\frac{\sqrt{15}}{3} \quad \text{or} \quad x=-1-\frac{\sqrt{15}}{3}$$

$$x=\frac{-3+\sqrt{15}}{3} \quad \text{or} \quad x=\frac{-3-\sqrt{15}}{3}$$

Check that the solution set is $\left\{\frac{-3\pm\sqrt{15}}{3}\right\}$.

Objective 5

Solve quadratic equations with solutions that are not real numbers.

CLASSROOM EXAMPLE 9 — **Solve for Nonreal Complex Solutions**

Solve the equation.

Solution:

$$x^2=-17$$

$$x=\sqrt{-17} \quad \text{or} \quad x=-\sqrt{-17}$$

$$x=i\sqrt{17} \quad \text{or} \quad x=-i\sqrt{17}$$

The solution set is $\left\{\pm i\sqrt{17}\right\}$.

CLASSROOM EXAMPLE 9 — **Solve for Nonreal Complex Solutions (cont'd)**

Solve the equation.

Solution:

$$(x+5)^2=-100$$

$$x+5=\sqrt{-100} \quad \text{or} \quad x+5=-\sqrt{-100}$$

$$x+5=10i \quad \text{or} \quad x+5=-10i$$

$$x=-5+10i \quad \text{or} \quad x=-5-10i$$

The solution set is $\{-5\pm10i\}$.

CLASSROOM EXAMPLE 9 — **Solve for Nonreal Complex Solutions (cont'd)**

Solve the equation.

Solution:

$$5x^2-15x+12=0$$

$$5x^2-15x=-12$$

$$x^2-3x=-\frac{12}{5}$$

Complete the square.

$$\left[\tfrac{1}{2}(-3)\right]^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4}$$

$$x^2-3x+\frac{9}{4}=-\frac{12}{5}+\frac{9}{4}$$

$$\left(x-\frac{3}{2}\right)^2=-\frac{3}{20}$$

CLASSROOM EXAMPLE 9 — **Solve for Nonreal Complex Solutions (cont'd)**

$$x-\frac{3}{2}=\sqrt{-\frac{3}{20}} \quad \text{or} \quad x-\frac{3}{2}=-\sqrt{-\frac{3}{20}}$$

$$x-\frac{3}{2}=\frac{i\sqrt{3}}{\sqrt{20}}\cdot\frac{\sqrt{5}}{\sqrt{5}} \quad \text{or} \quad x-\frac{3}{2}=\frac{-i\sqrt{3}}{\sqrt{20}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$$

$$x-\frac{3}{2}=\frac{i\sqrt{15}}{10} \quad \text{or} \quad x-\frac{3}{2}=\frac{-i\sqrt{15}}{10}$$

$$x=\frac{3}{2}+\frac{i\sqrt{15}}{10} \quad \text{or} \quad x=\frac{3}{2}-\frac{i\sqrt{15}}{10}$$

The solution set is $\left\{\frac{3}{2}\pm\frac{\sqrt{15}}{10}i\right\}$.