9.2 The Quadratic Formula

Objectives

1 Derive the quadratic formula.

2 Solve quadratic equations by using the quadratic formula.

3 Use the discriminant to determine the number and type of solutions.

PEARSON

Derive the quadratic formula.

Solve $ax^2 + bx + c = 0$ by completing the square (assuming $a > 0$).

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$or \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Derive the quadratic formula.

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad or \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad or \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad or \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad or \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Derive the quadratic formula.

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ ($a \neq 0$) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

CLASSROOM EXAMPLE 1 Using the Quadratic Formula (Rational Solutions)

Solve $4x^2 - 11x - 3 = 0$.

Solution:

$a = 4$, $b = -11$ and $c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(4)(-3)}}{2(4)}$$

$$x = \frac{11 \pm \sqrt{121 + 48}}{8}$$

$$x = \frac{11 \pm \sqrt{169}}{8}$$

$$x = \frac{11 \pm 13}{8}$$

$$x = \frac{11 + 13}{8}$$

$$= \frac{24}{8} = 3$$

$$x = \frac{11 - 13}{8}$$

$$= \frac{-2}{8} = -\frac{1}{4}$$

The solution set is $\left\{-\frac{1}{4}, 3\right\}$.

CLASSROOM EXAMPLE 2 Using the Quadratic Formula (Irrational Solutions)

Solve $2x^2 + 19 = 14x$.

Solution:

$$2x^2 - 14x + 19 = 0$$

$a = 2$, $b = -14$ and $c = 19$

$$x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(2)(19)}}{2(2)}$$

$$x = \frac{14 \pm \sqrt{196 - 152}}{4}$$

$$x = \frac{14 \pm \sqrt{44}}{4}$$

$$x = \frac{14 \pm \sqrt{4 \cdot 11}}{4}$$

$$x = \frac{14 + 2\sqrt{11}}{4}$$

$$= \frac{2(7 + \sqrt{11})}{4} = \frac{7 + \sqrt{11}}{2}$$

$$x = \frac{14 - 2\sqrt{11}}{4}$$

$$= \frac{2(7 - \sqrt{11})}{4} = \frac{7 - \sqrt{11}}{2}$$

The solution set is $\left\{\frac{7 \pm \sqrt{11}}{2}\right\}$.

CLASSROOM EXAMPLE 1

Solve quadratic equations by using the quadratic formula.

CAUTION

*1. **Every quadratic equation must be expressed in standard form ax2+bx+c=0 before we begin to solve it,*** whether we use factoring or the quadratic formula.

2. When writing solutions in lowest terms, be sure to FACTOR FIRST. Then divide out the common factor.

CLASSROOM EXAMPLE 3 — Using the Quadratic Formula (Nonreal Complex Solutions)

Solve $(x+5)(x+1)=10x.$

Solution:

$$x^2+6x+5=10x$$
$$x^2-4x+5=0$$

a = 1, *b* = –4 and *c* = 5

$$x=\frac{-b\pm\sqrt{(b)^2-4(a)(c)}}{2(a)}$$
$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$$
$$x=\frac{4\pm\sqrt{16-20}}{4}$$
$$x=\frac{4\pm\sqrt{-4}}{2}$$
$$x=\frac{4\pm2i}{2}$$
$$x=\frac{2(2\pm i)}{2}$$
$$x=2\pm i$$

The solution set is $\{2\pm i\}.$

Use the discriminant to determine the number and type of solutions.

Discriminant

The discriminant of $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If *a, b*, and *c* are integers, then the number and type of solutions are determined as follows.

Discriminant	*Number and Type of Solutions*
Positive, and the square of an integer	Two rational solutions
Positive, but not the square of an integer	Two irrational solutions
Zero	One rational solution
Negative	Two nonreal complex solutions

CLASSROOM EXAMPLE 4 — Using the Discriminant

Find the discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

$$10x^2-x-2=0$$

Solution:

a = 10, *b* = –1, *c* = –2

$$b^2-4ac=(-1)^2-4(10)(-2)$$
$$=1+80$$
$$=81$$

There will be two rational solutions, and the equation can be solved by factoring.

CLASSROOM EXAMPLE 4 — Using the Discriminant (cont'd)

Find each discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

$$3x^2-x=7$$

Solution:

$$3x^2-x-7=0$$
$$b^2-4ac=(-1)^2-4(3)(-7)$$
$$=1+84$$
$$=85$$

There will be two irrational solutions. Solve by using the quadratic formula.

$$16x^2+25=40x$$
$$16x^2-40x+25=0$$
$$a=16,\ b=-40,\ c=25$$
$$b^2-4ac=(-40)^2-4(16)(25)$$
$$=1600-1600$$
$$=0$$

There will be one rational solution. Solve by factoring.

CLASSROOM EXAMPLE 5 — Using the Discriminant

Find *k* so that the equation will have exactly one rational solution.
$x^2 - kx + 64 = 0$

Solution:

$$b^2-4ac=(-k)^2-4(1)(64)$$
$$=k^2-256$$
$$k^2-256=0$$
$$k^2=256$$
$$k=16\quad\text{or}\quad k=-16$$

There will be only on rational solution if *k* = 16 or *k* = –16.