9.3 Equations Quadratic in Form

Objectives

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.

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Equations Quadratic in Form

METHODS FOR SOLVING QUADRATIC EQUATIONS

Method	***Advantages***	***Disadvantages***
Factoring	This is usually the fastest method.	Not all polynomials are factorable; some factorable polynomials are difficult to factor.
Square root property	This is the simplest method for solving equations of the form $(ax+b)^2 = c$.	Few equations are given in this form.
Completing the square	This method can always be used, although most people prefer the quadratic formula.	It requires more steps than other methods.
Quadratic formula	This method can always be used.	It is more difficult than factoring because of the square root, although calculators can simplify its use.

Objective 1

Solve an equation with fractions by writing it in quadratic form.

CLASSROOM EXAMPLE 1 **Solving an Equation with Fractions that Leads to a Quadratic Equation**

Solve $\dfrac{4}{x-1}+9=-\dfrac{7}{x}$.

Solution:

Multiply by the LCD, $x(x-1)$.

$$x(x-1)\left(\frac{4}{x-1}+9\right)=x(x-1)\left(-\frac{7}{x}\right)$$
$$4x+9x(x-1)=-7(x-1)$$
$$4x+9x^2-9x=-7x+7$$
$$9x^2+2x-7=0$$

CLASSROOM EXAMPLE 1 **Solving an Equation with Fractions that Leads to a Quadratic Equation (cont'd)**

$$9x^2+2x-7=0$$
$$(9x-7)(x+1)=0$$
$$9x-7=0 \quad \text{or} \quad x+1=0$$
$$x=\frac{7}{9} \quad \text{or} \quad x=-1$$

The solution set is $\left\{-1,\dfrac{7}{9}\right\}$.

Objective 2

Use quadratic equations to solve applied problems.

CLASSROOM EXAMPLE 2 Solving a Motion Problem

In 1 ¾ hr Cody rows his boat 5 mi upriver and comes back. The rate of the current is 3 mph. How fast does Cody row?

Solution:

Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.** Let x = the speed Cody can row.
Make a table. Use $t = d/r$.

	d	*r*	*t*
Upstream	5	$x - 3$	$\frac{5}{x-3}$
Downstream	5	$x + 3$	$\frac{5}{x+3}$

Times in hours.

CLASSROOM EXAMPLE 2 Solving a Motion Problem (cont'd)

Step 3 **Write an equation.** The time going upriver added to the time going downriver is $1\frac{3}{4}$ or $\frac{7}{4}$ hr.

$$\frac{5}{x-3} + \frac{5}{x+3} = \frac{7}{4}$$

Step 4 **Solve** the equation. Multiply each side by the LCD, $4(x-3)(x+3)$.

$$4(x-3)(x+3)\frac{5}{x-3} + 4(x-3)(x+3)\frac{5}{x+3} = 4(x-3)(x+3)\left(\frac{7}{4}\right)$$

$$20(x+3) + 20(x-3) = 7(x-3)(x+3)$$

CLASSROOM EXAMPLE 2 Solving a Motion Problem (cont'd)

$$20x + 60 + 20x - 60 = 7\left(x^2 - 9\right)$$

$$40x = 7x^2 - 63$$

$$0 = 7x^2 - 40x - 63$$

$$0 = (7x+9)(x-7)$$

$$7x + 9 \quad \text{or} \quad x - 7$$

$$x = -\frac{9}{7} \quad \text{or} \quad x = 7$$

Step 5 **State the answer.** The speed cannot be negative, so Cody rows at the speed of 7mph.

Step 6 **Check** that this value satisfies the original problem.

Use quadratic equations to solve applied problems.

PROBLEM-SOLVING HINT

Recall from **Section 7.5** that a person's work rate is $\frac{1}{t}$ part of the job per hour, where t is the time in hours required to do the complete job. Thus, the part of the job the person will do in x hours is $\frac{1}{t}x$.

CLASSROOM EXAMPLE 3 Solving a Work Problem

Two chefs are preparing a banquet. One chef could prepare the banquet in 2 hr less time than the other. Together, they complete the job in 5 hr. How long would it take the faster chef working alone?

Solution:

Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.** Let x = the slow chef's time alone. Then, $x - 2$ = the fast chef's time alone.

	Rate	Time working Together	Fractional Part of the Job Done
Slow	$\frac{1}{x}$	5	$\frac{5}{x}$
Fast	$\frac{1}{x-2}$	5	$\frac{5}{x-2}$

CLASSROOM EXAMPLE 3 Solving a Work Problem (cont'd)

Step 3 **Write an equation.** Since together they complete 1 job,

$$\frac{5}{x} + \frac{5}{x-2} = 1.$$

Step 4 **Solve** the equation. Multiply each side by the LCD, $x(x-2)$.

$$x(x-2)\left(\frac{5}{x}\right) + x(x-2)\left(\frac{5}{x-2}\right) = x(x-2)(1)$$

$$5(x-2) + 5x = x(x-2)$$

$$5x - 10 + 5x = x^2 - 2x$$

$$0 = x^2 - 12x + 10$$

CLASSROOM EXAMPLE 3 Solving a Work Problem (cont'd)

Here $a = 1$, $b = -12$, and $c = 10$.

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(10)}}{2(1)}$$ **Use the quadratic formula.**

$$x = \frac{12 \pm \sqrt{144 - 40}}{2} = \frac{12 \pm \sqrt{104}}{2} = \frac{12 \pm 2\sqrt{26}}{2}$$

$$= \frac{2\left(6 \pm \sqrt{26}\right)}{2} = 6 \pm \sqrt{26}$$

CLASSROOM EXAMPLE 3 Solving a Work Problem (cont'd)

Step 5 **State the answer.** The slow chef's time cannot be 0.9 since the fast chef's time would then be 0.9 – 2 or – 1.1. So the slow chef's time working alone is 11.1 hr and the fast chef's time working alone is 11.1 – 2 = 9.1 hr.

Step 6 **Check** that this value satisfies the original problem.

Objective 3

Solve an equation with radicals by writing it in quadratic form.

CLASSROOM EXAMPLE 4 Solving Radical Equations That Lead to Quadratic Equations

Solve $2x = \sqrt{x} + 1$.

Solution:

$$2x - 1 = \sqrt{x}$$

$$(2x-1)^2 = \left(\sqrt{x}\right)^2$$ **Isolate.**

$$4x^2 - 4x + 1 = x$$ **Square.**

$$4x^2 - 5x + 1 = 0$$

$$(4x-1)(x-1) = 0$$

$$4x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{1}{4} \quad \text{or} \quad x = 1$$

CLASSROOM EXAMPLE 4 Solving Radical Equations That Lead to Quadratic Equations (cont'd)

Check both proposed solutions in the ***original*** equation,

$x = \frac{1}{4}$	or $x = 1$
$2x = \sqrt{x} + 1$	$2x = \sqrt{x} + 1$
$\frac{1}{2} = \frac{1}{2} + 1$	$2 = \sqrt{1} + 1$
	$2 = 2$
False	**True**

The solution set is $\{1\}$.

Objective 4

Solve an equation that is quadratic in form by substitution.

CLASSROOM EXAMPLE 5 **Defining Substitution Variables**

Define a variable *u*, and write each equation in the form $au^2 + bu + c = 0$.

$$2x^4 + 5x^2 - 12 = 0 \qquad 2(x+5)^2 - 7(x+5) + 6 = 0$$

Solution:

Let $u = x^2$. Let $u = (x + 5)$.

$$2u^2 + 5u - 12 = 0 \qquad 2u^2 - 7u + 6 = 0$$

CLASSROOM EXAMPLE 5 **Defining Substitution Variables (cont'd)**

Define a variable *u*, and write the equation in the form $au^2 + bu + c = 0$.

$$x^{\frac{4}{3}} - 8x^{\frac{2}{3}} + 16 = 0$$

Solution:

Let $u = x^{\frac{2}{3}}$.

$$u^2 - 8u + 16 = 0$$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form**

Solve $9x^4 - 37x^2 + 4 = 0.$

Solution:

Let $y = x^2$, so $y^2 = (x^2)^2 = x^4$

$$9y^2 - 37y + 4 = 0$$

$$(y-4)(9y-1) = 0$$

$$y - 4 = 0 \quad \text{or} \quad 9y - 1 = 0$$

$$y = 4 \quad \text{or} \quad y = \frac{1}{9}$$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

To find *x*, substitute x^2 for *y*.

$$x^2 = 4 \quad \text{or} \quad x^2 = \frac{1}{9}$$

$$x = \pm 2 \quad \text{or} \quad x = \pm\frac{1}{3}$$

Check

$$144 - 148 + 4 = 0 \qquad \frac{1}{9} - \frac{37}{9} + 4 = 0$$

$$0 = 0 \qquad 0 = 0$$

True **True**

The solution set is $\left\{\pm\frac{1}{3}, \pm 2\right\}$.

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

Solve $x^4 - 4x^2 = -2.$

Solution:

$$x^4 - 4x^2 + 2 = 0$$

Let $y = x^2$, so $y^2 = (x^2)^2 = x^4$.

***a* = 1, *b* = –4, *c* = 2**

$$y^2 - 4y + 2 = 0$$

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2} = \frac{2\left(2 \pm \sqrt{2}\right)}{2} = 2 \pm \sqrt{2}$$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

To find *x*, substitute x^2 for *y*.

$$x^2 = 2 \pm \sqrt{2}$$

$$x = \pm\sqrt{2 \pm \sqrt{2}}$$

Check

$$\left(2+\sqrt{2}\right)^2 - 4\left(2+\sqrt{2}\right) = -2 \qquad \left(2-\sqrt{2}\right)^2 - 4\left(2-\sqrt{2}\right) = -2$$

$$4 + 4\sqrt{2} + 2 - 8 - 4\sqrt{2} = -2 \qquad 4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} = -2$$

True $-2 = -2$ **True** $-2 = -2$

The solution set is $\left\{\pm\sqrt{2+\sqrt{2}}, \pm\sqrt{2-\sqrt{2}}\right\}$.

Solve an equation that is quadratic in form by substitution.

Solving an Equation That is Quadratic in Form by Substitution

Step 1 **Define a temporary variable *u*,** based on the relationship between the variable expressions in the given equation. Substitute *u* in the original equation and rewrite the equation in the form $au^2 + bu + c = 0$.

Step 2 **Solve the quadratic equation obtained in *Step 1*** by factoring or the quadratic formula.

Step 3 **Replace *u* with the expression it defined in *Step 1.***

Step 4 **Solve the resulting equations for the original variable.**

Step 5 **Check** all solutions by substituting them in the original equation.

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form**

Solve.

$$5(x+3)^2 + 9(x+3) = 2$$

Solution:

Let $y = x + 3$, so the equation becomes:

$$5y^2 - 9y = 2$$

$$(5y-1)(y+2) = 0$$

$$5y - 1 = 0 \quad \text{or} \quad y + 2 = 0$$

$$y = \frac{1}{5} \quad \text{or} \quad y = -2$$

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

To find *x*, substitute *x* + 3 for *y*.

$$x + 3 = \frac{1}{5} \quad \text{or} \quad x + 3 = -2$$

$$x = -\frac{14}{5} \quad \text{or} \quad x = -5$$

Check

$$\frac{1}{5} + \frac{9}{5} = 2 \qquad 20 - 18 = 2$$

$$2 = 2 \qquad 2 = 2$$

True **True**

The solution set is $\left\{-5, -\frac{14}{5}\right\}$.

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

Solve.

$$4x^{2/3} = 3x^{1/3} + 1$$

Solution:

Let $y = x^{1/3}$, so $y^2 = (x^{1/3})^2 = x^{2/3}$.

$$4y^2 = 3y + 1$$

$$4y^2 - 3y - 1 = 0$$

$$(4y+1) = 0 \quad \text{or} \quad (y-1) = 0$$

$$4y + 1 = 0 \quad \text{or} \quad y - 1 = 0$$

$$y = -\frac{1}{4} \quad \text{or} \quad y = 1$$

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

To find *x*, substitute $x^{1/3}$ for *y*.

$$x^{1/3} = -\frac{1}{4} \quad \text{or} \quad x^{1/3} = 1$$

$$(x^{1/3})^3 = \left(-\frac{1}{4}\right)^3 \quad \text{or} \quad (x^{1/3})^3 = (1)^3$$

$$x = -\frac{1}{64} \quad \text{or} \quad x = 1$$

Check

$$\frac{1}{4} = -\frac{3}{4} + 1 \qquad 4 = 3 + 1$$

$$\frac{1}{4} = \frac{1}{4} \qquad 4 = 4 \quad \textbf{True}$$

$2 = 2$ **True**

The solution set is $\left\{-\frac{1}{64}, 1\right\}$.