9.1 The Square Root Property and Completing the Square

Objectives

1 Review the zero-factor property.

2 Learn the square root property.

3 Solve quadratic equations of the form $(ax + b)^2 = c$ by extending the square root property.

4 Solve quadratic equations by completing the square.

5 Solve quadratic equations with solutions that are not real numbers.

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The Square Root Property and Completing the Square

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where *a, b*, and *c* are real numbers, with $a \neq 0$, is a **quadratic equation**. The given form is called **standard form**.

Review the zero-factor property.

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

CLASSROOM EXAMPLE 1 **Using the Zero-Factor Property**

Use the zero-factor property to solve $2x^2 - 3x = -1$.

Solution:

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$2x = 1 \qquad\qquad x = 1$$

$$x = \frac{1}{2}$$

Solution set is $\left\{\frac{1}{2}, 1\right\}$.

Learn the square root property.

Square Root Property

If *x* and *k* are complex numbers and $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

CAUTION Remember that if $k \neq 0$, using the square root property always produces ***two*** square roots, one positive and one negative.

CLASSROOM EXAMPLE 2 **Using the Zero-Factor Property**

Solve each equation.

Solution:

$$m^2 = 64$$

By the square root property, $m = 8$ or $m = -8$.

The solution set is {–8, 8}.

$$3x^2 - 54 = 0$$

$$3x^2 = 54$$

$$x^2 = 18$$

By the square root property,

$$x = \sqrt{18} \quad \text{or} \quad x = -\sqrt{18},$$

$$x = 3\sqrt{2} \quad \text{or} \quad x = -3\sqrt{2}.$$

Check:

$$x = \pm 3\sqrt{2}: \quad 3(18) - 54 = 0$$

True

Solution set is $\left\{3\sqrt{2}, -3\sqrt{2}\right\}$.

CLASSROOM EXAMPLE 3 — **Using the Square Root Property in an Application**

An expert marksman can hold a silver dollar at forehead level, drop it, draw his gun, and shoot the coin as it passes waist level. If the coin falls about 4 ft, use the formula $d = 16t^2$ to find the time that elapses between the dropping of the coin and the shot.

Solution:

$$d = 16t^2$$

$$4 = 16t^2$$

$$\frac{1}{4} = t^2$$

By the square root property,

$$t = \frac{1}{2} \quad \text{or} \quad t = -\frac{1}{2}$$

Since time cannot be negative, we discard the negative solution. Therefore, 0.5 sec elapses between the dropping of the coin and the shot.

CLASSROOM EXAMPLE 4 — **Extending the Square Root Property**

Solve $(x - 3)^2 = 16$.

Solution:

$$x - 3 = \sqrt{16} \quad \text{or} \quad x - 3 = -\sqrt{16}$$

$$x - 3 = 4 \qquad x - 3 = -4$$

$$x = 7 \qquad x = -1$$

CLASSROOM EXAMPLE 4 — **Extending the Square Root Property (cont'd)**

Check:

$$(7-3)^2 = 16 \qquad (-1-3)^2 = 16$$

$$4^2 = 16 \qquad -4^2 = 16$$

$$16 = 16 \qquad 16 = 16$$

True | **True**

The solution set is $\{-1, 7\}$.

CLASSROOM EXAMPLE 5 — **Extending the Square Root Property**

Solve $(3x + 1)^2 = 2$.

Solution:

$$3x + 1 = \sqrt{2} \quad \text{or} \quad 3x + 1 = -\sqrt{2}$$

$$3x = -1 + \sqrt{2} \qquad 3x = -1 - \sqrt{2}$$

$$x = \frac{-1 + \sqrt{2}}{3} \qquad x = \frac{-1 - \sqrt{2}}{3}$$

CLASSROOM EXAMPLE 5 — **Extending the Square Root Property (cont'd)**

We show a check for the first solution. The check for the other solution is similar.

Check:

$$(3x + 1)^2 = 2$$

$$\left[3\left(\frac{-1+\sqrt{2}}{3}\right)+1\right]^2 = 2 \quad ? \quad \textbf{Let } x = \frac{-1+\sqrt{2}}{3}.$$

$$\left(-1+\sqrt{2}+1\right)^2 = 2 \quad ? \quad \textbf{Multiply.}$$

$$\left(\sqrt{2}\right)^2 = 2 \quad ? \quad \textbf{Simplify.}$$

$$2 = 2 \quad \textbf{True}$$

The solution set is $\left\{\frac{-1 \pm \sqrt{2}}{3}\right\}$.

CLASSROOM EXAMPLE 6 — **Solving a Quadratic Equation by Completing the Square ($a = 1$)**

Solve $x^2 - 2x - 10 = 0$.

Solution:

$$x^2 - 2x = 10$$

Completing the square

$$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1.$$

Add 1 to each side.

$$x^2 - 2x + 1 = 10 + 1$$

$$(x - 1)^2 = 11$$

CLASSROOM EXAMPLE 6 Solving a Quadratic Equation by Completing the Square (a = 1) (cont'd)

Use the square root property.

$$x-1=\sqrt{11} \quad \text{or} \quad x-1=-\sqrt{11}$$
$$x=1+\sqrt{11} \quad \text{or} \quad x=1-\sqrt{11}$$

Check: $x=1+\sqrt{11}$:

$$\left(1+\sqrt{11}\right)^2-2\left(1+\sqrt{11}\right)-10=0 \quad ?$$
$$12+2\sqrt{11}-2-2\sqrt{11}-10=0 \quad \textbf{True}$$

The solution set is $\left\{1\pm\sqrt{11}\right\}$.

Solve quadratic equations by completing the square.

Completing the Square

To solve $ax^2 + bx + c = 0$ ($a \neq 0$) by completing the square, use these steps.

Step 1 **Be sure the second-degree (squared) term has coefficient 1.** If the coefficient of the squared term is one, proceed to ***Step 2.*** If the coefficient of the squared term is not 1 but some other nonzero number *a*, divide each side of the equation by *a*.

Step 2 **Write the equation in correct form** so that terms with variables are on one side of the equals symbol and the constant is on the other side.

Step 3 **Square half the coefficient of the first-degree (linear) term.**

Solve quadratic equations by completing the square.

Completing the Square (continued)

Step 4 **Add the square to each side.**

Step 5 **Factor the perfect square trinomial.** One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

Step 6 **Solve the equation.** Apply the square root property to complete the solution.

Steps 1 and 2 can be done in either order. With some equations, it is more convenient to do ***Step 2*** first.

CLASSROOM EXAMPLE 7 Solving a Quadratic Equation by Completing the Square (a = 1)

Solve $x^2 + 3x - 1 = 0$.

Solution:

$$x^2 + 3x = 1$$

Completing the square.

$$\left[\frac{1}{2}(3)\right]^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

Add the square to each side.

$$x^2 + 3x + \frac{9}{4} = 1 + \frac{9}{4}$$

$$\left(x+\frac{3}{2}\right)^2=\frac{13}{4}$$

CLASSROOM EXAMPLE 7 Solving a Quadratic Equation by Completing the Square (a = 1) (cont'd)

Use the square root property.

$$x+\frac{3}{2}=\sqrt{\frac{13}{4}} \quad \text{or} \quad x+\frac{3}{2}=-\sqrt{\frac{13}{4}}$$
$$x+\frac{3}{2}=\frac{\sqrt{13}}{2} \quad \text{or} \quad x+\frac{3}{2}=-\frac{\sqrt{13}}{2}$$
$$x=\frac{-3+\sqrt{13}}{2} \quad \text{or} \quad x=\frac{-3-\sqrt{13}}{2}$$

Check that the solution set is $\left\{\frac{-3\pm\sqrt{13}}{2}\right\}$.

CLASSROOM EXAMPLE 8 Solving a Quadratic Equation by Completing the Square ($a \neq 1$)

Solve $3x^2 + 6x - 2 = 0$.

Solution:

$$3x^2 + 6x = 2$$
$$x^2+2x=\frac{2}{3}$$

Completing the square.

$$\left[\tfrac{1}{2}(2)\right]^2=(1)^2=1$$
$$x^2+2x+1=\frac{2}{3}+1$$
$$(x+1)^2=\frac{5}{3}$$

CLASSROOM EXAMPLE 8 Solving a Quadratic Equation by Completing the Square ($a \neq 1$) (cont'd)

Use the square root property.

$$x+1=\sqrt{\frac{5}{3}} \quad \text{or} \quad x+1\frac{3}{2}=-\sqrt{\frac{5}{3}}$$

$$x=-1+\sqrt{\frac{5}{3}} \quad \text{or} \quad x=-1-\sqrt{\frac{5}{3}}$$

$$x=-1+\frac{\sqrt{15}}{3} \quad \text{or} \quad x=-1-\frac{\sqrt{15}}{3}$$

$$x=\frac{-3+\sqrt{15}}{3} \quad \text{or} \quad x=\frac{-3-\sqrt{15}}{3}$$

Check that the solution set is $\left\{\frac{-3\pm\sqrt{15}}{3}\right\}$.

Objective 5

Solve quadratic equations with solutions that are not real numbers.

CLASSROOM EXAMPLE 9 Solve for Nonreal Complex Solutions

Solve the equation.

Solution:

$$x^2=-17$$

$$x=\sqrt{-17} \quad \text{or} \quad x=-\sqrt{-17}$$

$$x=i\sqrt{17} \quad \text{or} \quad x=-i\sqrt{17}$$

The solution set is $\{\pm i\sqrt{17}\}$.

CLASSROOM EXAMPLE 9 Solve for Nonreal Complex Solutions (cont'd)

Solve the equation.

Solution:

$$(x+5)^2=-100$$

$$x+5=\sqrt{-100} \quad \text{or} \quad x+5=-\sqrt{-100}$$

$$x+5=10i \quad \text{or} \quad x+5=-10i$$

$$x=-5+10i \quad \text{or} \quad x=-5-10i$$

The solution set is $\{-5\pm 10i\}$.

CLASSROOM EXAMPLE 9 Solve for Nonreal Complex Solutions (cont'd)

Solve the equation.

Solution:

$$5x^2-15x+12=0$$

$$5x^2-15x=-12$$

$$x^2-3x=-\frac{12}{5}$$

Complete the square.

$$\left[\tfrac{1}{2}(-3)\right]^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4}$$

$$x^2-3x+\frac{9}{4}=-\frac{12}{5}+\frac{9}{4}$$

$$\left(x-\frac{3}{2}\right)^2=-\frac{3}{20}$$

CLASSROOM EXAMPLE 9 Solve for Nonreal Complex Solutions (cont'd)

$$x-\frac{3}{2}=\sqrt{-\frac{3}{20}} \quad \text{or} \quad x-\frac{3}{2}=-\sqrt{-\frac{3}{20}}$$

$$x-\frac{3}{2}=\frac{i\sqrt{3}}{\sqrt{20}}\cdot\frac{\sqrt{5}}{\sqrt{5}} \quad \text{or} \quad x-\frac{3}{2}=\frac{-i\sqrt{3}}{\sqrt{20}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$$

$$x-\frac{3}{2}=\frac{i\sqrt{15}}{10} \quad \text{or} \quad x-\frac{3}{2}=\frac{-i\sqrt{15}}{10}$$

$$x=\frac{3}{2}+\frac{i\sqrt{15}}{10} \quad \text{or} \quad x=\frac{3}{2}-\frac{i\sqrt{15}}{10}$$

The solution set is $\left\{\frac{3}{2}\pm\frac{\sqrt{15}}{10}i\right\}$.

9.2 The Quadratic Formula

Objectives

1. Derive the quadratic formula.
2. Solve quadratic equations by using the quadratic formula.
3. Use the discriminant to determine the number and type of solutions.

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Derive the quadratic formula.

Solve $ax^2 + bx + c = 0$ by completing the square (assuming $a > 0$).

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$or \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Derive the quadratic formula.

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad or \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad or \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad or \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad or \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Derive the quadratic formula.

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ ($a \neq 0$) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

CLASSROOM EXAMPLE 1 Using the Quadratic Formula (Rational Solutions)

Solve $4x^2 - 11x - 3 = 0.$

Solution:

$a = 4$, $b = -11$ and $c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(4)(-3)}}{2(4)}$$

$$x = \frac{11 \pm \sqrt{121 + 48}}{8}$$

$$x = \frac{11 \pm \sqrt{169}}{8}$$

$$x = \frac{11 \pm 13}{8}$$

$$x = \frac{11 + 13}{8} = \frac{24}{8} = 3$$

$$x = \frac{11 - 13}{8} = \frac{-2}{8} = -\frac{1}{4}$$

The solution set is $\left\{-\frac{1}{4}, 3\right\}$.

CLASSROOM EXAMPLE 2 Using the Quadratic Formula (Irrational Solutions)

Solve $2x^2 + 19 = 14x.$

Solution:

$$2x^2 - 14x + 19 = 0$$

$a = 2$, $b = -14$ and $c = 19$

$$x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$$

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(2)(19)}}{2(2)}$$

$$x = \frac{14 \pm \sqrt{196 - 152}}{4}$$

$$x = \frac{14 \pm \sqrt{44}}{4}$$

$$x = \frac{14 \pm \sqrt{4 \cdot 11}}{4}$$

$$x = \frac{14 + 2\sqrt{11}}{4} = \frac{2(7 + \sqrt{11})}{4} = \frac{7 + \sqrt{11}}{2}$$

$$x = \frac{14 - 2\sqrt{11}}{4} = \frac{2(7 - \sqrt{11})}{4} = \frac{7 - \sqrt{11}}{2}$$

The solution set is $\left\{\frac{7 \pm \sqrt{11}}{2}\right\}$.

CLASSROOM EXAMPLE 1

Solve quadratic equations by using the quadratic formula.

CAUTION

1. ***Every quadratic equation must be expressed in standard form $ax2+bx+c=0$ before we begin to solve it,*** whether we use factoring or the quadratic formula.
2. ***When writing solutions in lowest terms, be sure to FACTOR FIRST. Then divide out the common factor.***

CLASSROOM EXAMPLE 3 **Using the Quadratic Formula (Nonreal Complex Solutions)**

Solve $(x+5)(x+1)=10x.$

Solution:

$$x^2+6x+5=10x$$
$$x^2-4x+5=0$$

a = 1, *b* = –4 and *c* = 5

$$x=\frac{-b\pm\sqrt{(b)^2-4(a)(c)}}{2(a)}$$
$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$$
$$x=\frac{4\pm\sqrt{16-20}}{4}$$
$$x=\frac{4\pm\sqrt{-4}}{2}$$
$$x=\frac{4\pm 2i}{2}$$
$$x=\frac{2(2\pm i)}{2}$$
$$x=2\pm i$$

The solution set is $\{2\pm i\}.$

Use the discriminant to determine the number and type of solutions.

Discriminant

The discriminant of $ax^2+bx+c=0$ is $\mathbf{b^2-4ac}$. If *a*, *b*, and *c* are integers, then the number and type of solutions are determined as follows.

Discriminant	*Number and Type of Solutions*
Positive, and the square of an integer	Two rational solutions
Positive, but not the square of an integer	Two irrational solutions
Zero	One rational solution
Negative	Two nonreal complex solutions

CLASSROOM EXAMPLE 4 **Using the Discriminant**

Find the discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

$$10x^2-x-2=0$$

Solution:

a = 10, *b* = –1, *c* = –2

$$b^2-4ac=(-1)^2-4(10)(-2)$$
$$=1+80$$
$$=81$$

There will be two rational solutions, and the equation can be solved by factoring.

CLASSROOM EXAMPLE 4 **Using the Discriminant (cont'd)**

Find each discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

$$3x^2-x=7$$

Solution:

$$3x^2-x-7=0$$
$$b^2-4ac=(-1)^2-4(3)(-7)$$
$$=1+84$$
$$=85$$

There will be two irrational solutions. Solve by using the quadratic formula.

$$16x^2+25=40x$$
$$16x^2-40x+25=0$$

$a = 16,\ b = -40,\ c = 25$

$$b^2-4ac=(-40)^2-4(16)(25)$$
$$=1600-1600$$
$$=0$$

There will be one rational solution. Solve by factoring.

CLASSROOM EXAMPLE 5 **Using the Discriminant**

Find *k* so that the equation will have exactly one rational solution.

$x^2-kx+64=0$

Solution:

$$b^2-4ac=(-k)^2-4(1)(64)$$
$$=k^2-256$$
$$k^2-256=0$$
$$k^2=256$$
$$k=16\quad\text{or}\quad k=-16$$

There will be only on rational solution if *k* = 16 or *k* = –16.

9.3 Equations Quadratic in Form

Objectives

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.

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Equations Quadratic in Form

METHODS FOR SOLVING QUADRATIC EQUATIONS

Method	***Advantages***	***Disadvantages***
Factoring	This is usually the fastest method.	Not all polynomials are factorable; some factorable polynomials are difficult to factor.
Square root property	This is the simplest method for solving equations of the form $(ax+b)^2 = c$.	Few equations are given in this form.
Completing the square	This method can always be used, although most people prefer the quadratic formula.	It requires more steps than other methods.
Quadratic formula	This method can always be used.	It is more difficult than factoring because of the square root, although calculators can simplify its use.

Objective 1

Solve an equation with fractions by writing it in quadratic form.

CLASSROOM EXAMPLE 1 **Solving an Equation with Fractions that Leads to a Quadratic Equation**

Solve $\frac{4}{x-1}+9=-\frac{7}{x}$.

Solution:

Multiply by the LCD, $x(x-1)$.

$$x(x-1)\left(\frac{4}{x-1}+9\right)=x(x-1)\left(-\frac{7}{x}\right)$$
$$4x+9x(x-1)=-7(x-1)$$
$$4x+9x^2-9x=-7x+7$$
$$9x^2+2x-7=0$$

CLASSROOM EXAMPLE 1 **Solving an Equation with Fractions that Leads to a Quadratic Equation (cont'd)**

$$9x^2+2x-7=0$$
$$(9x-7)(x+1)=0$$
$$9x-7=0 \quad \text{or} \quad x+1=0$$
$$x=\frac{7}{9} \quad \text{or} \quad x=-1$$

The solution set is $\left\{-1, \frac{7}{9}\right\}$.

Objective 2

Use quadratic equations to solve applied problems.

CLASSROOM EXAMPLE 2 **Solving a Motion Problem**

In 1 ¾ hr Cody rows his boat 5 mi upriver and comes back. The rate of the current is 3 mph. How fast does Cody row?

Solution:

Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.** Let *x* = the speed Cody can row.

Make a table. Use *t* = *d/r*.

	d	*r*	*t*
Upstream	5	$x - 3$	$\frac{5}{x-3}$
Downstream	5	$x + 3$	$\frac{5}{x+3}$

Times in hours.

CLASSROOM EXAMPLE 2 **Solving a Motion Problem (cont'd)**

Step 3 **Write an equation.** The time going upriver added to the time going downriver is $1\frac{3}{4}$ or $\frac{7}{4}$ hr.

$$\frac{5}{x-3} + \frac{5}{x+3} = \frac{7}{4}$$

Step 4 **Solve** the equation. Multiply each side by the LCD, 4(*x* – 3)(*x* + 3).

$$4(x-3)(x+3)\frac{5}{x-3} + 4(x-3)(x+3)\frac{5}{x+3} = 4(x-3)(x+3)\left(\frac{7}{4}\right)$$

$$20(x+3) + 20(x-3) = 7(x-3)(x+3)$$

CLASSROOM EXAMPLE 2 **Solving a Motion Problem (cont'd)**

$$20x + 60 + 20x - 60 = 7\left(x^2 - 9\right)$$

$$40x = 7x^2 - 63$$

$$0 = 7x^2 - 40x - 63$$

$$0 = (7x+9)(x-7)$$

$$7x+9 \quad \text{or} \quad x-7$$

$$x = -\frac{9}{7} \quad \text{or} \quad x = 7$$

Step 5 **State the answer.** The speed cannot be negative, so Cody rows at the speed of 7mph.

Step 6 **Check** that this value satisfies the original problem.

Use quadratic equations to solve applied problems.

PROBLEM-SOLVING HINT

Recall from **Section 7.5** that a person's work rate is $\frac{1}{t}$ part of the job per hour, where *t* is the time in hours required to do the complete job. Thus, the part of the job the person will do in *x* hours is $\frac{1}{t}x$.

CLASSROOM EXAMPLE 3 **Solving a Work Problem**

Two chefs are preparing a banquet. One chef could prepare the banquet in 2 hr less time than the other. Together, they complete the job in 5 hr. How long would it take the faster chef working alone?

Solution:

Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.** Let *x* = the slow chef's time alone. Then, *x* – 2 = the fast chef's time alone.

	Rate	**Time working Together**	**Fractional Part of the Job Done**
Slow	$\frac{1}{x}$	5	$\frac{5}{x}$
Fast	$\frac{1}{x-2}$	5	$\frac{5}{x-2}$

CLASSROOM EXAMPLE 3 **Solving a Work Problem (cont'd)**

Step 3 **Write an equation.** Since together they complete 1 job,

$$\frac{5}{x} + \frac{5}{x-2} = 1.$$

Step 4 **Solve** the equation. Multiply each side by the LCD, *x*(*x* – 2).

$$x(x-2)\left(\frac{5}{x}\right) + x(x-2)\left(\frac{5}{x-2}\right) = x(x-2)(1)$$

$$5(x-2) + 5x = x(x-2)$$

$$5x - 10 + 5x = x^2 - 2x$$

$$0 = x^2 - 12x + 10$$

CLASSROOM EXAMPLE 3 **Solving a Work Problem (cont'd)**

Here *a* = 1, *b* = –12, and *c* = 10.

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(10)}}{2(1)}$$ **Use the quadratic formula.**

$$x = \frac{12 \pm \sqrt{144-40}}{2} = \frac{12 \pm \sqrt{104}}{2} = \frac{12 \pm 2\sqrt{26}}{2}$$

$$= \frac{2\left(6 \pm \sqrt{26}\right)}{2} = 6 \pm \sqrt{26}$$

CLASSROOM EXAMPLE 3 **Solving a Work Problem (cont'd)**

Step 5 **State the answer.** The slow chef's time cannot be 0.9 since the fast chef's time would then be 0.9 – 2 or – 1.1. So the slow chef's time working alone is 11.1 hr and the fast chef's time working alone is 11.1 – 2 = 9.1 hr.

Step 6 **Check** that this value satisfies the original problem.

Objective 3

Solve an equation with radicals by writing it in quadratic form.

CLASSROOM EXAMPLE 4 **Solving Radical Equations That Lead to Quadratic Equations**

Solve $2x = \sqrt{x} + 1$.

Solution:

$$2x - 1 = \sqrt{x}$$

$$(2x-1)^2 = \left(\sqrt{x}\right)^2 \quad \textbf{Isolate.}$$

$$4x^2 - 4x + 1 = x \quad \textbf{Square.}$$

$$4x^2 - 5x + 1 = 0$$

$$(4x-1)(x-1) = 0$$

$$4x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = \frac{1}{4} \quad \text{or} \quad x = 1$$

CLASSROOM EXAMPLE 4 **Solving Radical Equations That Lead to Quadratic Equations (cont'd)**

Check both proposed solutions in the ***original*** equation,

$x = \frac{1}{4}$	or $x = 1$
$2x = \sqrt{x} + 1$	$2x = \sqrt{x} + 1$
$\frac{1}{2} = \frac{1}{2} + 1$	$2 = \sqrt{1} + 1$
	$2 = 2$
False	**True**

The solution set is $\{1\}$.

Objective 4

Solve an equation that is quadratic in form by substitution.

CLASSROOM EXAMPLE 5 **Defining Substitution Variables**

Define a variable *u*, and write each equation in the form $au^2 + bu + c = 0$.

$2x^4 + 5x^2 - 12 = 0$ $2(x+5)^2 - 7(x+5) + 6 = 0$

Solution:

Let $u = x^2$. Let $u = (x + 5)$.

$2u^2 + 5u - 12 = 0$ $2u^2 - 7u + 6 = 0$

CLASSROOM EXAMPLE 5 **Defining Substitution Variables (cont'd)**

Define a variable *u*, and write the equation in the form $au^2 + bu + c = 0$.

$x^{\frac{4}{3}} - 8x^{\frac{2}{3}} + 16 = 0$

Solution:

Let $u = x^{\frac{2}{3}}$.

$u^2 - 8u + 16 = 0$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form**

Solve $9x^4 - 37x^2 + 4 = 0.$

Solution:

Let $y = x^2$, so $y^2 = (x^2)^2 = x^4$

$$9y^2 - 37y + 4 = 0$$

$$(y-4)(9y-1) = 0$$

$$y - 4 = 0 \quad \text{or} \quad 9y - 1 = 0$$

$$y = 4 \quad \text{or} \quad y = \frac{1}{9}$$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

To find *x*, substitute x^2 for *y*.

$$x^2 = 4 \quad \text{or} \quad x^2 = \frac{1}{9}$$

$$x = \pm 2 \quad \text{or} \quad x = \pm\frac{1}{3}$$

Check

$144 - 148 + 4 = 0$	$\frac{1}{9} - \frac{37}{9} + 4 = 0$
$0 = 0$	$0 = 0$
True	**True**

The solution set is $\left\{\pm\frac{1}{3}, \pm 2\right\}$.

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

Solve $x^4 - 4x^2 = -2.$

Solution:

$$x^4 - 4x^2 + 2 = 0$$

Let $y = x^2$, so $y^2 = (x^2)^2 = x^4$.

***a* = 1, *b* = –4, *c* = 2**

$$y^2 - 4y + 2 = 0$$

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2} = \frac{2\left(2 \pm \sqrt{2}\right)}{2} = 2 \pm \sqrt{2}$$

CLASSROOM EXAMPLE 6 **Solving Equations that Are Quadratic in Form (cont'd)**

To find *x*, substitute x^2 for *y*.

$$x^2 = 2 \pm \sqrt{2}$$

$$x = \pm\sqrt{2 \pm \sqrt{2}}$$

Check

$\left(2+\sqrt{2}\right)^2 - 4\left(2+\sqrt{2}\right) = -2$	$\left(2-\sqrt{2}\right)^2 - 4\left(2-\sqrt{2}\right) = -2$
$4 + 4\sqrt{2} + 2 - 8 - 4\sqrt{2} = -2$	$4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} = -2$
True $-2 = -2$	**True** $-2 = -2$

The solution set is $\left\{\pm\sqrt{2+\sqrt{2}}, \pm\sqrt{2-\sqrt{2}}\right\}$.

Solve an equation that is quadratic in form by substitution.

Solving an Equation That is Quadratic in Form by Substitution

Step 1 **Define a temporary variable *u*,** based on the relationship between the variable expressions in the given equation. Substitute *u* in the original equation and rewrite the equation in the form $au^2 + bu + c = 0$.

Step 2 **Solve the quadratic equation obtained in *Step 1*** by factoring or the quadratic formula.

Step 3 **Replace *u* with the expression it defined in *Step 1.***

Step 4 **Solve the resulting equations for the original variable.**

Step 5 **Check** all solutions by substituting them in the original equation.

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form**

Solve.

$$5(x+3)^2 + 9(x+3) = 2$$

Solution:

Let $y = x + 3$, so the equation becomes:

$$5y^2 - 9y = 2$$

$$(5y-1)(y+2) = 0$$

$$5y - 1 = 0 \quad \text{or} \quad y + 2 = 0$$

$$y = \frac{1}{5} \quad \text{or} \quad y = -2$$

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

To find *x*, substitute $x + 3$ for *y*.

$$x + 3 = \frac{1}{5} \quad \text{or} \quad x + 3 = -2$$

$$x = -\frac{14}{5} \quad \text{or} \quad x = -5$$

Check

$$\frac{1}{5} + \frac{9}{5} = 2 \qquad 20 - 18 = 2$$

$$2 = 2 \qquad 2 = 2$$

True **True**

The solution set is $\left\{-5, -\frac{14}{5}\right\}$.

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

Solve.

$$4x^{2/3} = 3x^{1/3} + 1$$

Solution:

Let $y = x^{1/3}$, so $y^2 = (x^{1/3})^2 = x^{2/3}$.

$$4y^2 = 3y + 1$$

$$4y^2 - 3y - 1 = 0$$

$$(4y+1) = 0 \quad \text{or} \quad (y-1) = 0$$

$$4y + 1 = 0 \quad \text{or} \quad y - 1 = 0$$

$$y = -\frac{1}{4} \quad \text{or} \quad y = 1$$

CLASSROOM EXAMPLE 7 **Solving Equations That Are Quadratic in Form (cont'd)**

To find *x*, substitute $x^{1/3}$ for *y*.

$$x^{1/3} = -\frac{1}{4} \quad \text{or} \quad x^{1/3} = 1$$

$$\left(x^{1/3}\right)^3 = \left(-\frac{1}{4}\right)^3 \quad \text{or} \quad \left(x^{1/3}\right)^3 = (1)^3$$

$$x = -\frac{1}{64} \quad \text{or} \quad x = 1$$

Check

$$\frac{1}{4} = -\frac{3}{4} + 1 \qquad 4 = 3 + 1$$

$$\frac{1}{4} = \frac{1}{4} \qquad 4 = 4 \quad \textbf{True}$$

$$2 = 2 \quad \textbf{True}$$

The solution set is $\left\{-\frac{1}{64}, 1\right\}$.

9.4 Formulas and Further Applications

Objectives

1 Solve formulas for variables involving squares and square roots.

2 Solve applied problems using the Pythagorean theorem.

3 Solve applied problems using area formulas.

4 Solve applied problems using quadratic functions as models.

PEARSON

Objective 1

Solve formulas for variables involving squares and square roots.

CLASSROOM EXAMPLE 1 **Solving for Variables Involving Squares or Square Roots**

Solve the formula for the given variable. Keep ± in the answer.

Solve $A = \pi r^2$ for *r*.

Solution:

$$\frac{A}{\pi} = r^2$$

$$r = \pm\sqrt{\frac{A}{\pi}}$$

$$r = \pm\frac{\sqrt{A}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$r = \pm\frac{\sqrt{A\pi}}{\pi}$$

CLASSROOM EXAMPLE 1 **Solving for Variables Involving Squares or Square Roots (cont'd)**

Solve the formula for the given variable.

Solve $s = 30\sqrt{\frac{a}{p}}$ for *a*.

Solution:

$$s^2 = 900 \cdot \frac{a}{p}$$ **Square both sides.**

$$ps^2 = 900a$$ **Multiply by *p*.**

$$\frac{ps^2}{900} = a$$ **Divide by 900.**

CLASSROOM EXAMPLE 2 **Solving for a Variable That Appears in First- and Second-Degree Terms**

Solve $2t^2 - 5t + k = 0$ for *t*.

Solution:

Use *a* = 2, *b* = –5, and *c* = *k* in the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)k}}{2(2)}$$

$$t = \frac{5 \pm \sqrt{25 - 8k}}{4}$$

The solutions are $t = \frac{5 + \sqrt{25 - 8k}}{4}$ and $t = \frac{5 - \sqrt{25 - 8k}}{4}$.

Objective 2

Solve applied problems using the Pythagorean theorem.

CLASSROOM EXAMPLE 3 — **Using the Pythagorean Theorem**

A ladder is leaning against a house. The distance from the bottom of the ladder to the house is 5 ft. The distance from the top of ladder to the ground is 1 ft less than the length of the ladder. How long is ladder?

Solution:

Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.**

Let x = the length of the ladder. Then, $x - 1$ = the distance from the top of the ladder to the ground.

CLASSROOM EXAMPLE 3 — **Using the Pythagorean Theorem (cont'd)**

Step 3 **Write an equation.**

The wall of the house is perpendicular to the ground, so this is a right triangle. Use the Pythagorean formula.

$$a^2 + b^2 = c^2$$

$$5^2 + (x-1)^2 = x^2$$

Step 4 **Solve.**

$$25 + x^2 - 2x + 1 = x^2$$

$$26 = 2x$$

$$13 = x$$

CLASSROOM EXAMPLE 3 — **Using the Pythagorean Theorem (cont'd)**

Step 5 **State the answer.**

The length of the ladder is 13 feet and the distance of the top of the ladder to the ground is 12 feet.

Step 6 **Check.**

$5^2 + 12^2 = 13^2$ and 12 is one less than 13, as required.

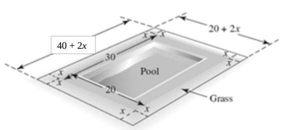
Objective 3

Solve applied problems using area formulas.

CLASSROOM EXAMPLE 4 — **Solving an Area Problem**

Suppose the pool is 20 ft by 40 ft. The homeowner wants to plant a strip of grass around the edge of the pool. There is enough seed to cover 700 ft². How wide should the grass strip be?

Solution:



Step 1 **Read** the problem carefully.

Step 2 **Assign the variable.**

Let x = the width of the grass strip

CLASSROOM EXAMPLE 4 — **Solving an Area Problem (cont'd)**

Step 3 **Write an equation.** The width of the larger rectangle is 20 + 20x, and the length is 40 + 2x.

Area of the rectangle – area of pool = area of grass

$$(20 + 2x)(40 + 2x) - 20(40) = 700$$

Step 4 **Solve.**

$$800 + 120x + 4x^2 - 800 = 700$$

$$4x^2 + 120x - 700 = 0$$

$$x^2 + 30x - 175 = 0$$

$$(x + 35)(x - 5) = 0$$

$$x + 35 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -35 \quad \text{or} \quad x = 5$$

CLASSROOM EXAMPLE 4 — **Solving an Area Problem (cont'd)**

Step 5 **State the answer.**

The width cannot be –35, so the grass strip should be 5 feet wide.

Step 6 **Check.**

If $x = 5$, then the area of the large rectangle is (40 + 2 • 5) = 50 • 30 = 1500 ft².

The area of the pool is 40 • 20 = 800ft².

So, the area of the grass strip is 1500 – 800 = 700 ft², as required. The answer is correct.

CLASSROOM EXAMPLE 5 — **Solving an Applied Problem Using a Quadratic Function**

A ball is projected upward from the ground. Its distance in feet from the ground at t seconds is $s(t) = -16t^2 + 64t$. At what time will the ball be 32 feet from the ground?

Solution:

$$s(t) = -16t^2 + 64t$$

$$32 = -16t^2 + 64t$$

$$16t^2 - 64t + 32 = 0$$

$$t^2 - 4t + 2 = 0$$

CLASSROOM EXAMPLE 5 — **Solving an Applied Problem Using a Quadratic Function (cont'd)**

Use $a = 1$, $b = -4$, and $c = 2$ in the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16-8}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

The solutions are $t = 2 + \sqrt{2} \approx 3.4 \text{ or } 2 - \sqrt{2} \approx 0.6.$

The ball will be at a height of 32 ft at about 0.6 seconds and 3.4 seconds.

CLASSROOM EXAMPLE 6 — **Using a Quadratic Function to Model the CPI**

The Consumer Price Index (CPI) is used to measure trends in prices for a "basket" of goods purchased by typical American families. This index uses a base year of 1967, which means that the index number for 1967 is 100. The quadratic function defined by

$$f(x) = -0.065x^2 + 14.8x + 249$$

approximates the CPI for the years 1980-2005, where x is the number of years that have elapsed since 1980.
(*Source:* Bureau of Labor Statistics.)

Use the model to approximate the CPI for 2000, to the nearest whole number.

In what year did the CPI reach 450? (Round down for the year.)

CLASSROOM EXAMPLE 6 — **Using a Quadratic Function to Model the CPI (cont'd)**

Use the model to approximate the CPI for 2000, to the nearest whole number.

$$f(x) = -0.065x^2 + 14.8x + 249$$

Solution:

For 2000, x = 2000 − 1980, so find *f* (20).

$$f(20) = -0.065(20)^2 + 14.8(20) + 249$$

$$f(20) = -26 + 296 + 249$$

$$f(20) \approx 519$$

The CPI for 2000 was about 519.

CLASSROOM EXAMPLE 6 — **Using a Quadratic Function to Model the CPI (cont'd)**

In what year did the CPI reach 450? (Round down for the year.)

$$f(x) = -0.065x^2 + 14.8x + 249$$

$$450 = -0.065x^2 + 14.8x + 249$$

$$0 = -0.065x^2 + 14.8x - 201$$

$$x = \frac{-14.8 \pm \sqrt{14.8^2 - 4(-0.065)(-201)}}{2(-0.065)}$$

$$x \approx 14.5 \text{ or } x \approx 213.2$$

The CPI first reached 450 in 1980 + 14yr = 1994. (The second solution is rejected as 1980 + 213 = 2192, which is far beyond period covered by the model.

9.5 Graphs of Quadratic Functions

Objectives

1. Graph a quadratic function.
2. Graph parabolas with horizontal and vertical shifts.
3. Use the coefficient of x^2 to predict the shape and direction in which a parabola opens.
4. Find a quadratic function to model data.

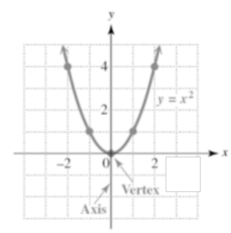
PEARSON

Graph a quadratic function.

The graph shown below is a graph of the simplest ***quadratic function,*** defined by $y = x^2$.

This graph is called a **parabola**.

x	y
−2	4
−1	1
0	0
1	1
2	4



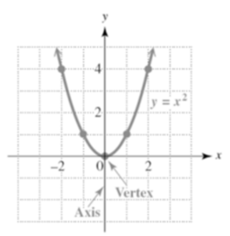
The point (0, 0), the lowest point on the curve, is the **vertex.**

Graph a quadratic function.

The vertical line through the vertex is the **axis** of the parabola, here $x = 0$.

A parabola is **symmetric about its axis**.

x	y
−2	4
−1	1
0	0
1	1
2	4



Graph parabolas with horizontal and vertical shifts.

Quadratic Function

A function that can be written in the form

$$f(x) = ax^2 + bx + c$$

for real numbers *a, b,* and *c*, with $a \neq 0$, is a **quadratic function.**

The graph of any quadratic function is a parabola with a vertical axis.

We use the variable ***y*** and function notation ***f (x)*** interchangeably. Although we use the letter ***f*** most often to name quadratic functions, other letters can be used. We use the capital letter ***F*** to distinguish between different parabolas graphed on the same coordinate axes.

Graph parabolas with horizontal and vertical shifts.

Parabolas do not need to have their vertices at the origin.

The graph of

$$F(x) = x^2 + k$$

is shifted, or translated k units vertically compared to $f(x) = x^2$.

CLASSROOM EXAMPLE 1 **Graphing a Parabola (Vertical Shift)**

Graph $f(x) = x^2 + 3$. Give the vertex, domain, and range.

Solution:

The graph has the same shape as $f(x) = x^2$, but shifted up 3 units.

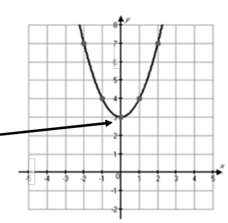
Make a table of points.

x	$x^2 + 3$
–2	7
–1	4
0	3
1	4
2	7

vertex (0, 3)

domain: $(-\infty, \infty)$

range: $[3, \infty)$



Graph parabolas with horizontal and vertical shifts.

Vertical Shift

The graph of $F(x) = x^2 + k$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The parabola is shifted k units up if $k > 0$, and $|k|$ units down if $k < 0$.
- The vertex is $(0, k)$.

CLASSROOM EXAMPLE 2 **Graphing a Parabola (Horizontal Shift)**

Graph $f(x) = (x + 2)^2$. Give the vertex, axis, domain, and range.

Solution:

The graph has the same shape as $f(x) = x^2$, but shifted 2 units to the left.

Make a table of points.

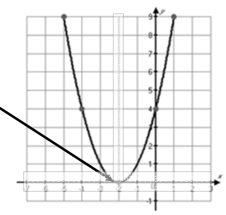
x	$(x + 2)^2$
–5	9
–4	4
–2	0
0	4
1	9

vertex (–2, 0)

axis $x = -2$

domain: $(-\infty, \infty)$

range: $[0, \infty)$



Graph parabolas with horizontal and vertical shifts.

Horizontal Shift

The graph of $F(x) = (x - h)^2$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The parabola is shifted h units to the right if $h > 0$, and $|h|$ units to the left if $h < 0$.
- The vertex is $(h, 0)$.

CLASSROOM EXAMPLE 3 **Graphing a Parabola (Horizontal and Vertical Shifts)**

Graph $f(x) = (x - 2)^2 + 1$. Give the vertex, axis, domain, and range.

Solution:

The graph has the same shape as $f(x) = x^2$, but shifted 2 units to the right and 3 unit up.

Make a table of points.

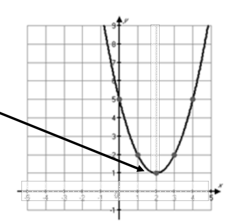
x	$f(x)$
0	5
1	2
2	1
3	2
4	5

vertex (2, 1)

axis $x = 2$

domain: $(-\infty, \infty)$

range: $[1, \infty)$



Graph parabolas with horizontal and vertical shifts.

Vertex and Axis of Parabola

The graph of $F(x) = (x - h)^2 + k$ is a parabola.

- The graph has the same shape as the graph of $f(x) = x^2$.
- The vertex of the parabola is (h, k).
- The axis is the vertical line $x = h$.

Objective 3

Use the coefficient of x^2 to predict the shape and direction in which a parabola opens.

CLASSROOM EXAMPLE 4

Graphing a Parabola That Opens Down

Graph $f(x) = -2x^2 - 3$. Give the vertex, axis, domain, and range.

Solution:

The coefficient (–2) affects the shape of the graph; the 2 makes the parabola narrower.

The negative sign makes the parabola open down.

The graph is shifted down 3 units.

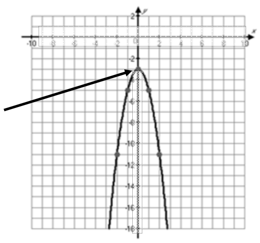
CLASSROOM EXAMPLE 4

Graphing a Parabola That Opens Down (cont'd)

Graph $f(x) = -2x^2 - 3$.

x	$f(x)$
–2	–11
–1	–5
0	–3
1	–
2	–11

vertex (0, –3)



axis $x = 0$ domain: $(-\infty, \infty)$ range: $(-\infty, -3]$

Use the coefficient of x^2 to predict the shape and direction in which a parabola opens.

General Principles of $F(x) = a(x - h)^2 + k$ $(a \neq 0)$

1. The graph of the quadratic function defined by

$$F(x) = a(x - h)^2 + k,\ a \neq 0,$$

is a parabola with vertex (h, k) and the vertical line $x = h$ as axis.

2. The graph opens up if a is positive and down if a is negative.

3. The graph is wider than that of $f(x) = x^2$ if $0 < |a| < 1$.
The graph is narrower than that of $f(x) = x^2$ if $|a| > 1$.

CLASSROOM EXAMPLE 5

Using the General Characteristics to Graph a Parabola

Graph $f(x) = \frac{1}{2}(x-2)^2 + 1.$

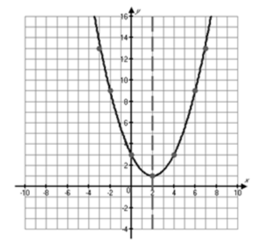
Solution:

Parabola opens up.
Narrower than $f(x) = x^2$
Vertex: (2, 1)

axis $x = 2$

domain: $(-\infty, \infty)$

range: $[1, \infty)$



9.6 More About Parabolas and Their Applications

Objectives

1 Find the vertex of a vertical parabola.

2 Graph a quadratic function.

3 Use the discriminant to find the number of x-intercepts of a parabola with a vertical axis.

4 Use quadratic functions to solve problems involving maximum or minimum value.

5 Graph parabolas with horizontal axes.

PEARSON

Find the vertex of a vertical parabola.

When the equation of a parabola is given in the form

$$f(x) = ax^2 + bx + c,$$

we need to locate the vertex to sketch an accurate graph.

There are two ways to do this:

1. Complete the square.
2. Use a formula derived by completing the square.

CLASSROOM EXAMPLE 1 **Completing the Square to Find the Vertex (a = 1)**

Find the vertex of the graph of $f(x) = x^2 + 4x - 9$.

Solution:

We need to complete the square.

$$f(x) = x^2 + 4x - 9$$
$$= (x^2 + 4x \qquad) - 9$$
$$\left[\tfrac{1}{2}(4)\right]^2 = 2^2 = 4$$
$$= (x^2 + 4x + 4 - 4) - 9$$
$$= (x^2 + 4x + 4) - 4 - 9$$
$$= (x + 2)^2 - 13$$

The vertex of the parabola is (–2, –13).

CLASSROOM EXAMPLE 2 **Completing the Square to Find the Vertex (a ≠ 1)**

Find the vertex of the graph of $f(x) = 2x^2 - 4x + 1$.

Solution:

We need to complete the square, factor out 2 from the first two terms.

$$f(x) = 2x^2 - 4x + 1$$
$$f(x) = 2(x^2 - 2x) + 1$$
$$= 2(x^2 - 2x + 1 - 1) + 1$$
$$\left[\tfrac{1}{2}(-2)\right]^2 = (-1)^2 = 1$$
$$= 2(x^2 - 2x + 1) + 2(-1) + 1$$
$$= 2(x^2 - 2x + 1) - 2 + 1$$
$$= 2(x - 1)^2 - 1$$

The vertex of the parabola is (1, –1).

Find the vertex of a vertical parabola.

Vertex Formula

The graph of the quadratic function defined by

$f(x) = ax^2 + bx + c$ ($a \neq 0$) has vertex $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$,

and the axis of the parabola is the line $x = \dfrac{-b}{2a}$.

CLASSROOM EXAMPLE 3 **Using the Formula to Find the Vertex**

Use the vertex formula to find the vertex of the graph of $f(x) = -2x^2 + 3x - 1$.

Solution:

$a = -2$, $b = 3$, and $c = -1$.

The x-coordinate: $\dfrac{-b}{2a} = \dfrac{-3}{2(-2)} = \dfrac{-3}{-4} = \dfrac{3}{4}$

The y-coordinate: $f\left(\tfrac{3}{4}\right) = -2\left(\tfrac{3}{4}\right)^2 + 3\left(\tfrac{3}{4}\right) - 1$

$$= -2\left(\tfrac{9}{16}\right) + \left(\tfrac{9}{4}\right) - 1$$
$$= -\tfrac{9}{8} + \tfrac{18}{8} - \tfrac{8}{8} = \tfrac{1}{8}$$

The vertex of the parabola is $\left(\dfrac{3}{4}, \dfrac{1}{8}\right)$.

Graph a quadratic function.

Graphing a Quadratic Function $y = f(x)$

Step 1 **Determine whether the graph opens up or down.** If $a > 0$, the parabola opens up; if $a < 0$, it opens down.

Step 2 **Find the vertex.** Use either the vertex formula or completing the square.

Step 3 **Find any intercepts.** To find the x-intercepts (if any), solve $f(x) = 0$. To find the y-intercept, evaluate $f(0)$.

Step 4 **Complete the graph.** Plot the points found so far. Find and plot additional points as needed, using symmetry about the axis.

CLASSROOM EXAMPLE 4 **Graphing a Quadratic Function**

Graph $f(x) = x^2 - 6x + 5$. Give the vertex, axis, domain, and range.
Solution:

Step 1 The graph opens up since $a = 1$, which is > 0.

Step 2 **Find the vertex.** Complete the square.

$$f(x) = x^2 - 6x + 5$$
$$= x^2 - 6x + 9 - 9 + 5$$
$$= (x - 3)^2 - 4$$

The vertex is at (3, –4).

CLASSROOM EXAMPLE 4 **Graphing a Quadratic Function (cont'd)**

Graph $f(x) = x^2 - 6x + 5$.

Step 3 **Find any x-intercepts.** Let $f(x) = 0$

$$0 = x^2 - 6x + 5$$
$$0 = (x - 5)(x - 1)$$
$$x - 5 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 5 \quad \text{or} \quad x = 1$$

The x-intercepts are (5, 0) and (1, 0).

Find the y-intercept. Let $x = 0$.

The y-intercept is (0, 5).

$$f(x) = 0^2 - 6(0) + 5 \quad = 5$$

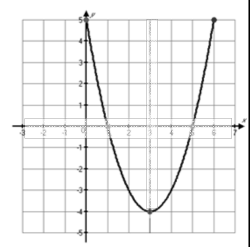
CLASSROOM EXAMPLE 4 **Graphing a Quadratic Function (cont'd)**

Graph $f(x) = x^2 - 6x + 5$.

Step 4 **Plot the points.**

Vertex: (3, –4)
x-intercepts: (5, 0) and (1, 0)
y-intercept. (0, 5)
axis of symmetry: $x = 3$

By symmetry (6, 5) is also another point on the graph.



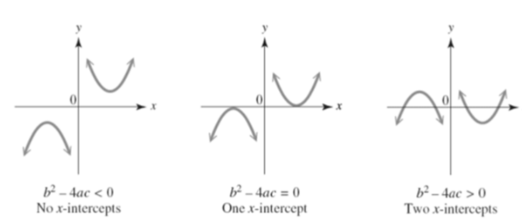
domain: $(-\infty, \infty)$ range: $[-4, \infty)$

Objective 3

Use the discriminant to find the number of *x*-intercepts of a parabola with a vertical axis.

Use the discriminant to find the number of *x*-intercepts of a parabola with a vertical axis.

You can use the discriminant to determine the number x-intercepts.



CLASSROOM EXAMPLE 5 — **Using the Discriminant to Determine the Number of *x*- Intercepts**

Find the discriminant and use it to determine the number of *x*-intercepts of the graph of the quadratic function.

$f(x) = -3x^2 - x + 2$

Solution:

$a = -3$, $b = -1$, $c = 2$

$$b^2 - 4ac = (-1)^2 - 4(-3)(2)$$
$$= 1 + 24$$
$$= 25$$

The discriminant is positive, the graph has two *x*-intercepts.

CLASSROOM EXAMPLE 5 — **Using the Discriminant to Determine the Number of *x*- Intercepts (cont'd)**

Find the discriminant and use it to determine the number of *x*-intercepts of the graph of each quadratic function.

$f(x) = x^2 - x + 1$

Solution:

$a = 1$, $b = -1$, $c = 1$

$$b^2 - 4ac = (-1)^2 - 4(1)(1)$$
$$= 1 - 4$$
$$= -3$$

The discriminant is negative, the graph has no *x*-intercepts.

$f(x) = x^2 - 8x + 16$

$a = 1$, $b = -8$, $c = 16$

$$b^2 - 4ac = (-8)^2 - 4(1)(16)$$
$$= 64 - 64$$
$$= 0$$

The discriminant is 0, the graph has one *x*-intercept.

Use quadratic functions to solve problems involving maximum or minimum value.

PROBLEM-SOLVING HINT

In many applied problems we must find the greatest or least value of some quantity. When we can express that quantity in terms of a quadratic function, the value of *k* in the vertex (*h, k*) gives that optimum value.

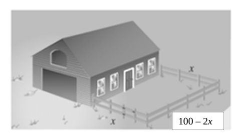
CLASSROOM EXAMPLE 6 — **Finding the Maximum Area of a Rectangular Region**

A farmer has 100 ft of fencing to enclose a rectangular area next to a building. Find the maximum area he can enclose, and the dimensions of the field when the area is maximized.

Solution:

Let *x* = the width of the field

x + *x* + length = 100
2*x* + length = 100
length = 100 – 2*x*



CLASSROOM EXAMPLE 6 — **Finding the Maximum Area of a Rectangular Region (cont'd)**

Area = width × length

$A(x) = x(100 - 2x)$
$= 100x - 2x^2$

Determine the vertex:

$a = -2$, $b = 100$, $c = 0$

$$x = \frac{-b}{2a} = \frac{-100}{2(-2)} = 25$$

y-coordinate: $f(25) = -2(25)^2 + 100(25)$
$= -2(625) + 2500$
$= -1250 + 2500$
$= 1250$

Vertex: (25, 1250)

CLASSROOM EXAMPLE 6 — **Finding the Maximum Area of a Rectangular Region (cont'd)**

Parabola opens down with vertex (25, 1250).

The vertex shows that the maximum area will be 1250 square feet.

The area will occur if the width, *x,* is 25 feet and the length is 100 – 2*x,* or 50 feet.

CLASSROOM EXAMPLE 7 — **Finding the Maximum Height Attained by a Projectile**

A toy rocket is launched from the ground so that its distance above the ground after t seconds is

$$s(t) = -16t^2 + 208t$$

Find the maximum height it reaches and the number of seconds it takes to reach that height.

Solution:

Find the vertex of the function.

$a = -16$, $b = 208$

$$x = \frac{-b}{2a} = \frac{-208}{2(-16)} = \frac{13}{2} = 6.5$$

CLASSROOM EXAMPLE 7 — **Finding the Maximum Height Attained by a Projectile (cont'd)**

Find the y-coordinate.

$$\begin{aligned} f\left(\tfrac{13}{2}\right) &= -16\left(\tfrac{13}{2}\right)^2 + 208\left(\tfrac{13}{2}\right) \\ &= -16\left(\tfrac{169}{4}\right) + 1352 \\ &= -676 + 1352 \\ &= 676 \end{aligned}$$

The toy rocket reaches a maximum height of 676 feet in 6.5 seconds.

Graph parabolas with horizontal axes.

Graph of a Horizontal Parabola

The graph of $x = ay^2 + by + c$ or $x = a(y - k)^2 + h$ is a parabola.

- The vertex of the parabola is (h, k).
- The axis is the horizontal line $y = k$.
- The graph opens to the right if $a > 0$ and to the left if $a < 0$.

CLASSROOM EXAMPLE 8 — **Graphing a Horizontal Parabola (a = 1)**

Graph $x = (y + 1)^2 - 4$. Give the vertex, axis, domain, and range.

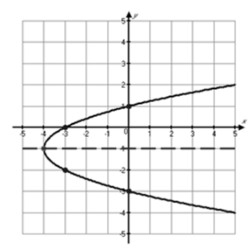
Solution:

Vertex: $(-4, -1)$

Opens: right since $a > 1$

Axis: $y = -1$

x	$f(x)$
–3	0
–3	–2
0	1
0	–3



Domain: $[-4, \infty)$ Range: $(-\infty, \infty)$

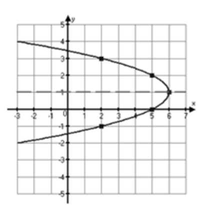
CLASSROOM EXAMPLE 9 — **Completing the Square to Graph a Horizontal Parabola ($a \neq 1$)**

Graph $x = -y^2 + 2y + 5$. Give the vertex, axis, domain, and range.

Solution:

Complete the square.

$$\begin{aligned} x &= -(y^2 - 2y) + 5 \\ &= -(y^2 - 2y + 1) - (-1) + 5 \\ &= -(y - 1)^2 + 6 \end{aligned}$$



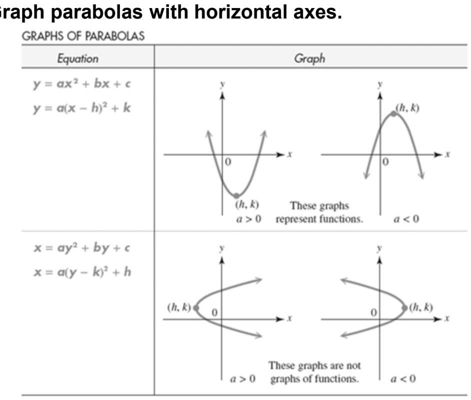
Vertex: $(6, 1)$

Opens: left, since $a < 1$

Axis: $y = 1$

Domain: $[-\infty, 6]$ Range: $(-\infty, \infty)$

Graph parabolas with horizontal axes.



9.7 Polynomial and Rational Inequalities

Objectives

1. Solve quadratic inequalities.
2. Solve polynomial inequalities of degree 3 or greater.
3. Solve rational inequalities.

Solve quadratic inequalities.

Quadratic Inequality

A **quadratic inequality** can be written in the form

$$ax^2 + bx + c < 0 \text{ or } ax^2 + bx + c > 0,$$

$$ax^2 + bx + c \le 0 \text{ or } ax^2 + bx + c \ge 0,$$

where a, b, and c are real numbers, with $a \neq 0$.

CLASSROOM EXAMPLE 1 **Solving Quadratic Inequalities by Graphing**

Use the graph to solve each quadratic inequality.

$x^2 + 6x + 8 > 0$

Solution:

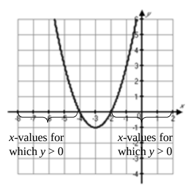
Find the x-intercepts.

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

$x + 2 = 0$ or $x + 4 = 0$

$x = -2$ or $x = -4$



Notice from the graph that x-values less than –4 or greater than –2 result in y-values ***greater** than* 0.

The solution set is $(-\infty, -4) \cup (-2, \infty)$.

CLASSROOM EXAMPLE 1 **Solving Quadratic Inequalities by Graphing (cont'd)**

Use the graph to solve each quadratic inequality.

$x^2 + 6x + 8 < 0$

Solution:

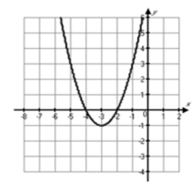
Find the x-intercepts.

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

$x + 2 = 0$ or $x + 4 = 0$

$x = -2$ or $x = -4$



Notice from the graph that x-values between –4 and –2 result in y-values ***less** than* 0.

The solution set is $(-4, -2)$.

CLASSROOM EXAMPLE 2 **Solving a Quadratic Inequality Using Test Numbers**

Solve and graph the solution set.

$2x^2 + 3x \ge 2$

Solution:

Use factoring to solve the quadratic equation.

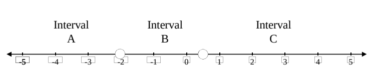
$$2x^2 + 3x - 2 = 0$$

$$(2x - 1)(x + 2) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = -2$$

The numbers divide a number line into three intervals.



CLASSROOM EXAMPLE 2 **Solving a Quadratic Inequality Using Test Numbers (cont'd)**

$2x^2 + 3x \ge 2$

Choose a number from each interval to substitute in the inequality.

Interval A: Let $x = -3$.

$2(-3)^2 + 3(-3) \ge 2$

$18 - 9 \ge 2$

$9 \ge 2$ **True**

Interval B: Let $x = 0$.

$2(0)^2 + 3(0) \ge 2$

$0 + 0 \ge 2$

$0 \ge 2$ **False**

CLASSROOM EXAMPLE 2

Solving a Quadratic Inequality Using Test Numbers (cont'd)

Interval C: Let x = 1. $2x^2 + 3x \geq 2$

$2(1)^2 + 3(1) \geq 2$

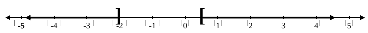
$2 + 3 \geq 2$

$5 \geq 2$ **True**

The numbers in Intervals A and C are solutions. The numbers –2 and ½ are included because of the ≥.

Solution set:

$$(-\infty, -2] \cup [\tfrac{1}{2}, \infty)$$



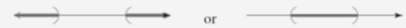
Solve quadratic inequalities.

Solving a Quadratic Inequality

Step 1 **Write the inequality as an equation and solve it.**

Step 2 **Use the solutions from Step 1 to determine intervals.** Graph the numbers found in Step 1 on a number line. These numbers divide the number line into intervals.

Step 3 **Find the intervals that satisfy the inequality.** Substitute a test number from each interval into the original inequality to determine the intervals that satisfy the inequality. All numbers in those intervals are in the solution set. A graph of the solution set will usually look like one of these. (Square brackets might be used instead of parentheses.)



Step 4 **Consider the endpoints separately.** The numbers from Step 1 are included in the solution set if the inequality symbol is ≤ or ≥; they are not included if it is < or >.

CLASSROOM EXAMPLE 3

Solving Special Cases

Solve.

Solution:

$(3x – 2)^2 > –2$

The square of any real number is always greater than or equal to 0, so any real number satisfies this inequality. The solution set is the set of all real numbers, (–∞, ∞).

$(3x – 2)^2 < –2$

The square of a real number is never negative, there is no solution for this inequality. The solution set is ∅.

Objective 2

Solve polynomial inequalities of degree 3 or greater.

CLASSROOM EXAMPLE 4

Solving a Third-Degree Polynomial Inequality

Solve and graph the solution set.

$(2x + 1)(3x – 1)(x + 4) > 0$

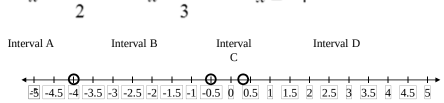
Solution:

Set each factored polynomial ***equal*** to 0 and solve the equation.

$(2x + 1)(3x – 1)(x + 4) = 0$

$2x + 1 = 0$ or $3x – 1 = 0$ or $x + 4 = 0$

$x = -\frac{1}{2}$ $\quad x = \frac{1}{3}$ $\quad x = -4$



CLASSROOM EXAMPLE 4

Solving a Third-Degree Polynomial Inequality (cont'd)

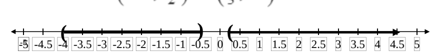
Substitute a test number from each interval in the ***original*** inequality.

Interval	Test Number	Test of Inequality	True or False?
A	–5	–144 > 0	False
B	–2	42 > 0	True
C	0	–4 > 0	False
D	1	30 > 0	True

The numbers in Intervals B and D, not including the endpoints are solutions.

Solution set:

$$(-4, \tfrac{-1}{2}) \cup (\tfrac{1}{3}, \infty)$$



Objective 3

Solve rational inequalities.

Solve rational inequalities.

Solving a Rational Inequality

Step 1 **Write the inequality so that 0 is on one side** and there is a single fraction on the other side.

Step 2 **Determine the numbers that make the numerator or denominator equal to 0.**

Step 3 **Divide a number line into intervals.** Use the numbers from ***Step 2.***

Step 4 **Find the intervals that satisfy the inequality.** Test a number from each interval by substituting it into the ***original*** inequality.

Step 5 **Consider the endpoints separately.** Exclude any values that make the denominator 0.

CLASSROOM EXAMPLE 5 **Solving a Rational Inequality**

Solve and graph the solution set.

$$\frac{2}{x-4} < 3$$

Solution:

Write the inequality so that 0 is on one side.

$$\frac{2}{x-4} - 3 < 0$$

$$\frac{2}{x-4} - \frac{3(x-4)}{x-4} < 0$$

$$\frac{2-3x+12}{x-4} < 0$$

$$\frac{-3x+14}{x-4} < 0$$

The number 14/3 makes the numerator 0, and 4 makes the denominator 0. These two numbers determine three intervals.

CLASSROOM EXAMPLE 5 **Solving a Rational Inequality (cont'd)**

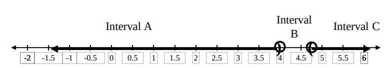
$$\frac{2}{x-4} < 3$$

Test a number from each interval.

Interval	Test Number	Test of Inequality	True or False?
A	0	-1/2 < 3	True
B	13/3	6 < 3	False
C	5	2 < 3	True

The solution set includes numbers in Intervals A and C, excluding endpoints.

Solution set: $(-\infty, 4) \cup \left(\frac{14}{3}, \infty\right)$



CLASSROOM EXAMPLE 6 **Solving a Rational Inequality**

Solve and graph the solution set.

$$\frac{x+2}{x-1} \le 5$$

Solution:

Write the inequality so that 0 is on one side.

$$\frac{x+2}{x-1} - 5 \le 0$$

$$\frac{x+2}{x-1} - \frac{5(x-1)}{x-1} \le 0$$

$$\frac{x+2-5x+5}{x-1} \le 0$$

$$\frac{-4x+7}{x-1} \le 0$$

The number 7/4 makes the numerator 0, and 1 makes the denominator 0. These two numbers determine three intervals.

CLASSROOM EXAMPLE 6 **Solving a Rational Inequality (cont'd)**

$$\frac{x+2}{x-1} \le 5$$

Test a number from each interval.

Interval	Test Number	Test of Inequality	True or False?
A	0	–2 ≤ 5	True
B	3/2	7 ≤ 5	False
C	2	4 ≤ 5	True

The numbers in Intervals A and C are solutions. 1 is NOT in the solution set (since it makes the denominator 0), but 7/4 is.

Solution set: $(-\infty, 1) \cup \left[\frac{7}{4}, \infty\right)$