
Example 1

Evaluate the limit $\lim_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$ using

(a) algebraic manipulation (factor and cancel)

Solution

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+4}{x+3} = \frac{7}{6}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} \overset{H}{=} \lim_{x\to 3} \frac{2x+1}{2x} = \frac{7}{6}$$

Example 2

Evaluate the limit $\lim_{x\to 0} \dfrac{\sin 3x}{\tan 4x}$ using

(a) the basic trigonometric limit $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ together with appropriate changes of variables

Solution

Write the limit as

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)\left(\lim_{x\to 0} \frac{x\cos 4x}{\sin 4x}\right)$$

In the first limit let $u = 3x$ and in the second let $v = 4x$. Then the limit is

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} &= \left(\lim_{u\to 0} \frac{3\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v\cos v}{4\sin v}\right)\\ &= \tfrac{3}{4}\left(\lim_{u\to 0} \frac{\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v}{\sin v}\right)\left(\lim_{v\to 0} \cos v\right) = \tfrac{3}{4}\end{aligned}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} \overset{H}{=} \lim_{x\to 0} \frac{3\cos 3x}{4\sec^2 4x} = \tfrac{3}{4}$$

Example 3

Evaluate the limit $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x$ using L'Hopital's Rule.

Solution
Write the limit as

$$\lim_{x \to \frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right) \tan x = \lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cot x}$$

Then direct substitution gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x \to \frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right) \tan x \stackrel{\text{H}}{=} \lim_{x \to \frac{\pi}{2}} \frac{1}{\left(-\csc^2 x\right)} = -1$$

Example 4

Evaluate the limit $\lim_{x \to 1} \dfrac{\sqrt{2-x} - x}{x - 1}$ using L'Hopital's Rule.

Solution
Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x \to 1} \frac{\sqrt{2-x} - x}{x - 1} \stackrel{\text{H}}{=} \lim_{x \to 1} \frac{-\dfrac{1}{\sqrt{2-x}} - 1}{1} = -\frac{3}{2}$$

Note that this result can also be obtained by rationalizing the numerator by multiplying top and bottom by the root conjugate $\sqrt{2-x} + x$.

Example 5

Evaluate the limit $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$ using L'Hopital's Rule.

Solution
Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \stackrel{\text{H}}{=} \lim_{x \to 0} \frac{\sin x}{2x}$$

Again direct substitution gives $\frac{0}{0}$ so use L'Hopital's Rule a second time to give

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \stackrel{\text{H}}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Example 6

Evaluate the limits at infinity

(a) $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\lim\limits_{x\to\infty} x^2 e^{-x}$

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) &\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}
\end{aligned}$$

Example 7

Evaluate the limit $\lim\limits_{x\to 2^+} \dfrac{\ln(x-2)}{\ln(x^2-4)}$ using L'Hopital's Rule.

Solution

Direct substitution gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\begin{aligned}
\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} &\overset{H}{=} \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{\frac{2x}{x^2-4}} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{x^2-4}{2x^2-4x} \\
&\overset{H}{=} \lim_{x\to 2^+} \frac{2x}{4x-4} = 1
\end{aligned}$$

Example 8

Evaluate the limit $\lim\limits_{x\to\infty}(\ln x - x)$ using L'Hopital's Rule.

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty}(\ln x - x) = \lim_{x\to\infty}\ln\left(xe^{-x}\right)$$

Then, as in Example 6 (a) and (b),

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Now let $u = xe^{-x}$, then $x \to \infty \Rightarrow u \to 0^+$. Hence

$$\lim_{x\to\infty}(\ln x - x) = \lim_{u\to 0^+}\ln u = -\infty$$

Example 9

Evaluate the limit $\lim\limits_{x\to 0^+}(\sin x)^{\sqrt{x}}$ using L'Hopital's Rule.

Solution

Direct substitution gives the indeterminate form 0^0. First find the natural logarithm of the limit, as

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = \lim_{x\to 0^+}\sqrt{x}\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\dfrac{1}{\sqrt{x}}}$$

Now direct substitution gives $\frac{\infty}{\infty}$, so we can use L'Hopital's Rule

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(\sin x)}{\dfrac{1}{\sqrt{x}}} &\overset{H}{=} \lim_{x\to 0^+}\frac{\dfrac{\cos x}{\sin x}}{\left(-\frac{1}{2}\right)x^{-3/2}} \\ &= -2\lim_{x\to 0^+}\frac{x^{3/2}\cos x}{\sin x} = -2\left(\lim_{x\to 0^+}\sqrt{x}\cos x\right)\left(\lim_{x\to 0^+}\frac{x}{\sin x}\right) = 0\end{aligned}$$

since

$$\lim_{x\to 0^+}\sqrt{x}\cos x = 0$$

and we can use L'Hopital's Rule on the second limit to give

$$\lim_{x\to 0^+}\frac{x}{\sin x} \overset{H}{=} \lim_{x\to 0^+}\frac{1}{\cos x} = 1$$

Hence

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = 0$$

so that

$$\lim_{x\to 0^+}(\sin x)^{\sqrt{x}} = e^0 = 1$$

7.7 Indeterminate Forms and L'Hospital's Rule

A Click here for answers.

S Click here for solutions.

1–45 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, use it. If l'Hospital's Rule doesn't apply, explain why.

1. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

2. $\lim_{x\to 1} \dfrac{x^2+3x-4}{x-1}$

3. $\lim_{x\to -1} \dfrac{x^6-1}{x^4-1}$

4. $\lim_{x\to 0} \dfrac{\tan x}{x+\sin x}$

5. $\lim_{x\to 0} \dfrac{e^x-1}{\sin x}$

6. $\lim_{x\to 0} \dfrac{x+\tan x}{\sin x}$

7. $\lim_{x\to 0} \dfrac{\sin x}{x^3}$

8. $\lim_{x\to \pi} \dfrac{\tan x}{x}$

9. $\lim_{x\to 3\pi/2} \dfrac{\cos x}{x-(3\pi/2)}$

10. $\lim_{t\to 16} \dfrac{\sqrt[4]{t}-2}{t-16}$

11. $\lim_{x\to a} \dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}$

12. $\lim_{x\to 0} \dfrac{6^x-2^x}{x}$

13. $\lim_{x\to \infty} \dfrac{(\ln x)^3}{x^2}$

14. $\lim_{x\to 0} \dfrac{\sin x}{e^x}$

15. $\lim_{x\to 0} \dfrac{\tan \alpha x}{x}$

16. $\lim_{x\to 0} \dfrac{\sin^2 x}{\tan(x^2)}$

17. $\lim_{x\to \infty} \dfrac{\ln \ln x}{\sqrt{x}}$

18. $\lim_{x\to \infty} \dfrac{\ln(1+e^x)}{5x}$

19. $\lim_{x\to 0} \dfrac{\tan^{-1}(2x)}{3x}$

20. $\lim_{x\to 0} \dfrac{x}{\sin^{-1}(3x)}$

21. $\lim_{x\to 0} \dfrac{\sin mx}{\sin nx}$

22. $\lim_{x\to 0} \dfrac{\sin^{10} x}{\sin(x^{10})}$

23. $\lim_{x\to 0} \dfrac{x+\sin 3x}{x-\sin 3x}$

24. $\lim_{x\to 0} \dfrac{e^{4x}-1}{\cos x}$

25. $\lim_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$

26. $\lim_{x\to 0} \dfrac{x+\tan 2x}{x-\tan 2x}$

27. $\lim_{x\to 0} \dfrac{\tan 2x}{\tanh 3x}$

28. $\lim_{x\to 0} \dfrac{2x-\sin^{-1}x}{2x+\cos^{-1}x}$

29. $\lim_{x\to 0} \dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$

30. $\lim_{x\to -\infty} xe^x$

31. $\lim_{x\to \infty} e^{-x}\ln x$

32. $\lim_{x\to (\pi/2)^-} \sec 7x \cos 3x$

33. $\lim_{x\to 0^+} \sqrt{x}\sec x$

34. $\lim_{x\to \pi} (x-\pi)\cot x$

35. $\lim_{x\to 1^+} (x-1)\tan(\pi x/2)$

36. $\lim_{x\to 0} \left(\dfrac{1}{x^4}-\dfrac{1}{x^2}\right)$

37. $\lim_{x\to \infty} \left(x-\sqrt{x^2-1}\right)$

38. $\lim_{x\to \infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

39. $\lim_{x\to \infty} \left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

40. $\lim_{x\to \infty} \left(\dfrac{x^3}{x^2-1}-\dfrac{x^3}{x^2+1}\right)$

41. $\lim_{x\to 0^+} (\sin x)^{\tan x}$

42. $\lim_{x\to \infty} \left(1+\dfrac{1}{x^2}\right)^x$

43. $\lim_{x\to 0^+} (\cot x)^{\sin x}$

44. $\lim_{x\to \infty} \left(1+\dfrac{1}{x}\right)^{x^2}$

45. $\lim_{x\to 0^-} (-\ln x)^x$

Answers

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{4}$ **2.** 5

3. $\frac{3}{2}$ **4.** $\frac{1}{2}$

5. 1 **6.** 2

7. ∞ **8.** 0

9. 1 **10.** $\frac{1}{32}$

11. $\frac{1}{3a^{2/3}}$ **12.** $\ln 3$

13. 0 **14.** 0

15. α **16.** 1

17. 0 **18.** $\frac{1}{5}$

19. $\frac{2}{3}$ **20.** $\frac{1}{3}$

21. $\frac{m}{n}$ **22.** 1

23. -2 **24.** 0

25. $\frac{1}{2}$ **26.** -3

27. $\frac{2}{3}$ **28.** 0

29. $\frac{1}{3}$ **30.** 0

31. 0 **32.** $\frac{3}{7}$

33. 0 **34.** 1

35. $-\frac{2}{\pi}$ **36.** ∞

37. 0 **38.** 1

39. 0 **40.** 0

41. 1 **42.** 1

43. 1 **44.** ∞

45. 1

Solutions

E Click here for exercises.

A Click here for answers.

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$.

1. $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}=\lim\limits_{x\to 2}\dfrac{x-2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\dfrac{1}{x+2}=\dfrac{1}{4}$

2. $\lim\limits_{x\to 1}\dfrac{x^2+3x-4}{x-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+4)}{x-1}=\lim\limits_{x\to 1}(x+4)=5$

3. $\lim\limits_{x\to -1}\dfrac{x^6-1}{x^4-1}\overset{\text{H}}{=}\lim\limits_{x\to -1}\dfrac{6x^5}{4x^3}=\dfrac{-6}{-4}=\dfrac{3}{2}$

4. $\lim\limits_{x\to 0}\dfrac{\tan x}{x+\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\sec^2 x}{1+\cos x}=\dfrac{1}{1+1}=\dfrac{1}{2}$

5. $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{\cos x}=\dfrac{1}{1}=1$

6. $\lim\limits_{x\to 0}\dfrac{x+\tan x}{\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1+\sec^2 x}{\cos x}=\dfrac{1+1^2}{1}=2$

7. $\lim\limits_{x\to 0}\dfrac{\sin x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{3x^2}=\infty$

8. $\lim\limits_{x\to \pi}\dfrac{\tan x}{x}=\dfrac{\tan\pi}{\pi}=\dfrac{0}{\pi}=0$. L'Hospital's Rule does not apply because the denominator doesn't approach 0.

9. $\lim\limits_{x\to 3\pi/2}\dfrac{\cos x}{x-3\pi/2}\overset{\text{H}}{=}\lim\limits_{x\to 3\pi/2}\dfrac{-\sin x}{1}=-\sin\frac{3\pi}{2}=1$

10.
$$\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{t-16}=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{\left(\sqrt{t}+4\right)\left(\sqrt{t}-4\right)}=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{\left(\sqrt{t}+4\right)\left(\sqrt[4]{t}+2\right)\left(\sqrt[4]{t}-2\right)}=\lim_{t\to 16}\frac{1}{\left(\sqrt{t}+4\right)\left(\sqrt[4]{t}+2\right)}=\frac{1}{(4+4)(2+2)}=\frac{1}{32}$$

11. $\lim\limits_{x\to a}\dfrac{x^{1/3}-a^{1/3}}{x-a}\overset{\text{H}}{=}\lim\limits_{x\to a}\dfrac{(1/3)\,x^{-2/3}}{1}=\dfrac{1}{3a^{2/3}}$

12. $\lim\limits_{x\to 0}\dfrac{6^x-2^x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{6^x(\ln 6)-2^x(\ln 2)}{1}=\ln 6-\ln 2=\ln\frac{6}{2}=\ln 3$

13.
$$\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{2x}=\lim_{x\to\infty}\frac{3(\ln x)^2}{2x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{6(\ln x)(1/x)}{4x}=\lim_{x\to\infty}\frac{3\ln x}{2x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3/x}{4x}=\lim_{x\to\infty}\frac{3}{4x^2}=0$$

14. $\lim\limits_{x\to 0}\dfrac{\sin x}{e^x}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

15. $\lim\limits_{x\to 0}\dfrac{\tan\alpha x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\alpha\sec^2\alpha x}{1}=\alpha$

16.
$$\lim_{x\to 0}\frac{\sin^2 x}{\tan\left(x^2\right)}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2\sin x\cos x}{2x\sec^2\left(x^2\right)}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\cos x}{\sec^2\left(x^2\right)}=1\cdot 1=1$$

17. $\lim\limits_{x\to\infty}\dfrac{\ln\ln x}{\sqrt{x}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/(x\ln x)}{1/\left(2\sqrt{x}\right)}=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{x}\ln x}=0$

18.
$$\lim_{x\to\infty}\frac{\ln\left(1+e^x\right)}{5x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x/\left(1+e^x\right)}{5}=\lim_{x\to\infty}\frac{e^x}{5\left(1+e^x\right)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{5e^x}=\frac{1}{5}$$

19. $\lim\limits_{x\to 0}\dfrac{\tan^{-1}(2x)}{3x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{2/\left(1+4x^2\right)}{3}=\dfrac{2}{3}$

20.
$$\lim_{x\to 0}\frac{x}{\sin^{-1}(3x)}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1}{3\Big/\sqrt{1-(3x)^2}}=\lim_{x\to 0}\frac{1}{3}\sqrt{1-9x^2}=\frac{1}{3}$$

21. $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{m\cos mx}{n\cos nx}=\dfrac{m}{n}$

22.
$$\lim_{x\to 0}\frac{\sin^{10}x}{\sin\left(x^{10}\right)}\overset{\text{H}}{=}\lim_{x\to 0}\frac{10\sin^9 x\cos x}{10x^9\cos\left(x^{10}\right)}=\left[\lim_{x\to 0}\frac{\sin x}{x}\right]^9\lim_{x\to 0}\frac{\cos x}{\cos\left(x^{10}\right)}=1^9\cdot 1=1$$

23. $\lim\limits_{x\to 0}\dfrac{x+\sin 3x}{x-\sin 3x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1+3\cos 3x}{1-3\cos 3x}=\dfrac{1+3}{1-3}=-2$

24. $\lim\limits_{x\to 0}\dfrac{e^{4x}-1}{\cos x}=\dfrac{0}{1}=0$

25.
$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}=\frac{0+2+1}{6}=\frac{1}{2}$$

26.
$$\lim_{x\to 0}\frac{x+\tan 2x}{x-\tan 2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1+2\sec^2 2x}{1-2\sec^2 2x}=\frac{1+2(1)^2}{1-2(1)^2}=-3$$

27. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{\tanh 3x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{2\sec^2 2x}{3\,\text{sech}^2\,3x}=\dfrac{2}{3}$

28. $\lim\limits_{x\to 0}\dfrac{2x-\sin^{-1}x}{2x+\cos^{-1}x}=\dfrac{2(0)-0}{2(0)+\pi/2}=0$. L'Hospital's Rule does not apply.

29. $\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2-1/\sqrt{1-x^2}}{2+1/(1+x^2)} = \frac{2-1}{2+1} = \frac{1}{3}$

30. $\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty}\frac{x}{e^{-x}} \overset{\text{H}}{=} \lim_{x\to -\infty}\frac{1}{-e^{-x}} = \lim_{x\to -\infty} -e^x = 0$

31. $\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{e^x} = \lim_{x\to\infty}\frac{1}{xe^x} = 0$

32. $\lim_{x\to(\pi/2)^-}\sec 7x\cos 3x = \lim_{x\to(\pi/2)^-}\frac{\cos 3x}{\cos 7x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-}\frac{-3\sin 3x}{-7\sin 7x} = \frac{3(-1)}{7(-1)} = \frac{3}{7}$

33. $\lim_{x\to 0^+}\sqrt{x}\sec x = 0\cdot 1 = 0$

34. $\lim_{x\to\pi}(x-\pi)\cot x = \lim_{x\to\pi}\frac{x-\pi}{\tan x} \overset{\text{H}}{=} \lim_{x\to\pi}\frac{1}{\sec^2 x} = \frac{1}{(-1)^2} = 1$

35. $\lim_{x\to 1^+}(x-1)\tan(\pi x/2) = \lim_{x\to 1^+}\frac{x-1}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+}\frac{1}{-\csc^2(\pi x/2)\,\frac{\pi}{2}} = -\frac{2}{\pi}$

36. $\lim_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right) = \lim_{x\to 0}\frac{1-x^2}{x^4} = \infty$

37. $\lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)$

$$= \lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{x^2-\left(x^2-1\right)}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}} = 0$$

38. $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

$$= \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{\left(x^2+x+1\right)-\left(x^2-x\right)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2+1/x}{\sqrt{1+1/x+1/x^2}+\sqrt{1-1/x}} = \frac{2}{1+1} = 1$$

39. $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{e^x-1}$ (since both limits exist) $= 0-0 = 0$

40. $\lim_{x\to\infty}\left(\frac{x^3}{x^2-1}-\frac{x^3}{x^2+1}\right) = \lim_{x\to\infty}\frac{x^3\left(x^2+1\right)-x^3\left(x^2-1\right)}{\left(x^2-1\right)\left(x^2+1\right)} = \lim_{x\to\infty}\frac{2x^3}{x^4-1} = \lim_{x\to\infty}\frac{2/x}{1-1/x^4} = 0$

41. $y = (\sin x)^{\tan x} \Rightarrow \ln y = \tan x\ln(\sin x)$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\tan x\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{(\cos x)/\sin x}{-\csc^2 x} = \lim_{x\to 0^+}(-\sin x\cos x) = 0 \Rightarrow$$

$$\lim_{x\to 0^+}(\sin x)^{\tan x} = \lim_{x\to 0^+}e^{\ln y} = e^0 = 1.$$

42. Let $y = \left(1+\frac{1}{x^2}\right)^x$. Then $\ln y = x\ln\left(1+\frac{1}{x^2}\right) \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}x\ln\left(1+\frac{1}{x^2}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\left(-\frac{2}{x^3}\right)\Big/\left(1+\frac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{2/x}{1+1/x^2} = 0,$$

so $\lim_{x\to\infty}\left(1+1/x^2\right)^x = \lim_{x\to\infty}e^{\ln y} = e^0 = 1$.

43. $y = (\cot x)^{\sin x} \Rightarrow \ln y = \sin x\ln(\cot x) \Rightarrow$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(\cot x)}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{\left(-\csc^2 x\right)/\cot x}{-\csc x\cot x} = \lim_{x\to 0^+}\frac{\csc x}{\cot^2 x} = \lim_{x\to 0^+}\frac{\sin x}{\cos^2 x} = 0$$

so $\lim_{x\to 0^+}(\cot x)^{\sin x} = \lim_{x\to 0^+}e^{\ln y} = e^0 = 1$.

44. Let $y = (1+1/x)^{x^2}$. Then $\ln y = x^2\ln(1+1/x) \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}x^2\ln(1+1/x) = \lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\left(-1/x^2\right)/(1+1/x)}{-2/x^3} = \lim_{x\to\infty}\frac{x}{2(1+1/x)} = \infty \Rightarrow$$

$$\lim_{x\to\infty}(1+1/x)^{x^2} = \lim_{x\to\infty}e^{\ln y} = \infty.$$

45. $y = (-\ln x)^x \Rightarrow \ln y = x\ln(-\ln x)$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}x\ln(-\ln x) = \lim_{x\to 0^+}\frac{\ln(-\ln x)}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{(1/-\ln x)(-1/x)}{-1/x^2} = \lim_{x\to 0^+}\frac{-x}{\ln x} = 0 \Rightarrow$$

$$\lim_{x\to 0^+}(-\ln x)^x = e^0 = 1.$$