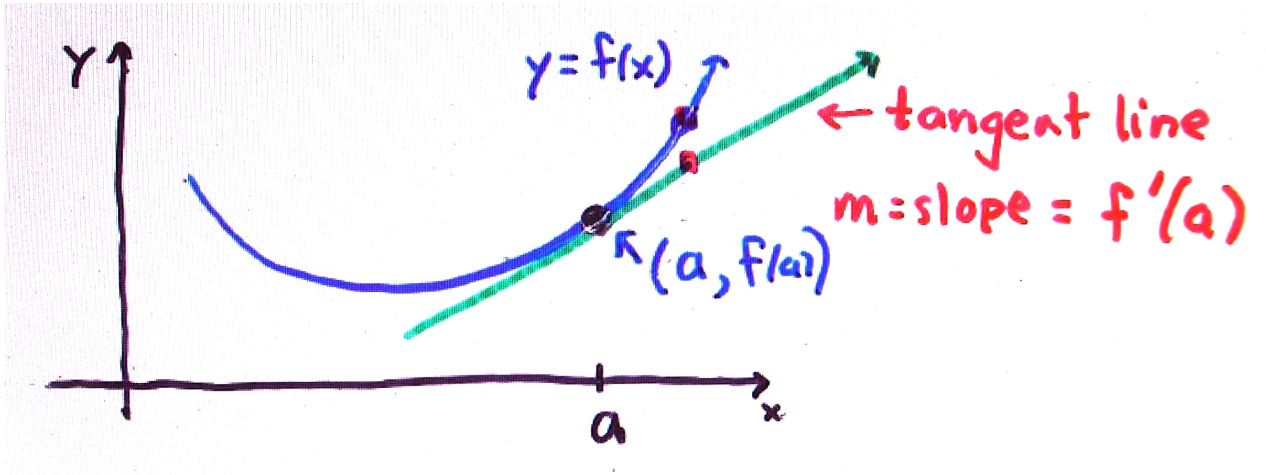
# Equation of Tangent Line

Recall the equation of the tangent line of a curve $y = f(x)$ at the point $x = a$.




The **general equation of the tangent line** is

$$y = L_a(x) := f(a) + f'(a)(x - a).$$

That is the point-slope form of a line through the point $(a, f(a))$ with slope $f'(a)$.

## Linear Approximation

It follows from the geometric picture as well as the equation

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

which means that $\frac{f(x)-f(a)}{x-a} \approx f'(a)$ or

$$f(x) \approx f(a) + f'(a)(x - a) = L_a(x)$$

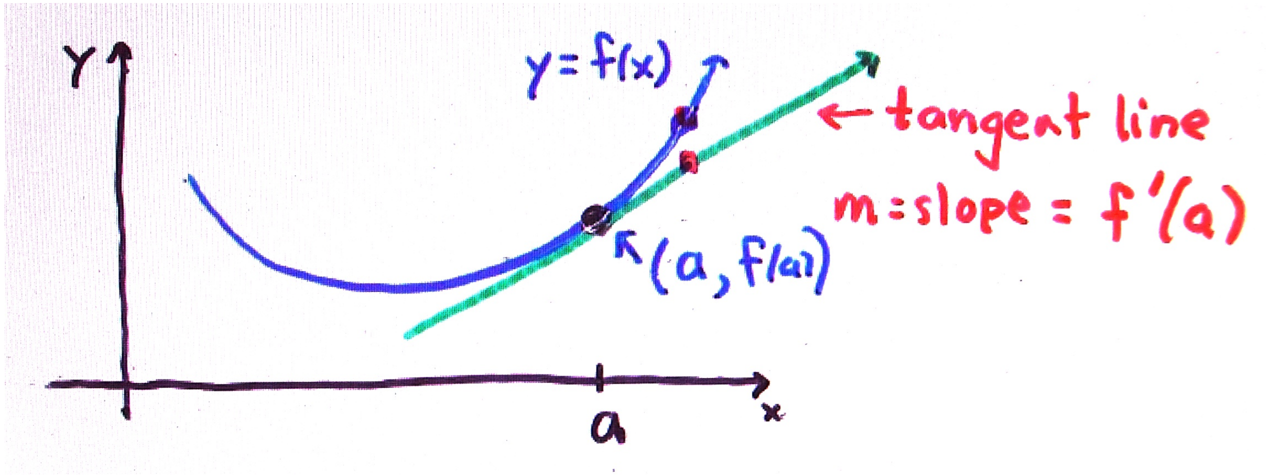
for $x$ close to $a$. Thus $L_a(x)$ is a good **approximation** of $f(x)$ for $x$ near $a$.

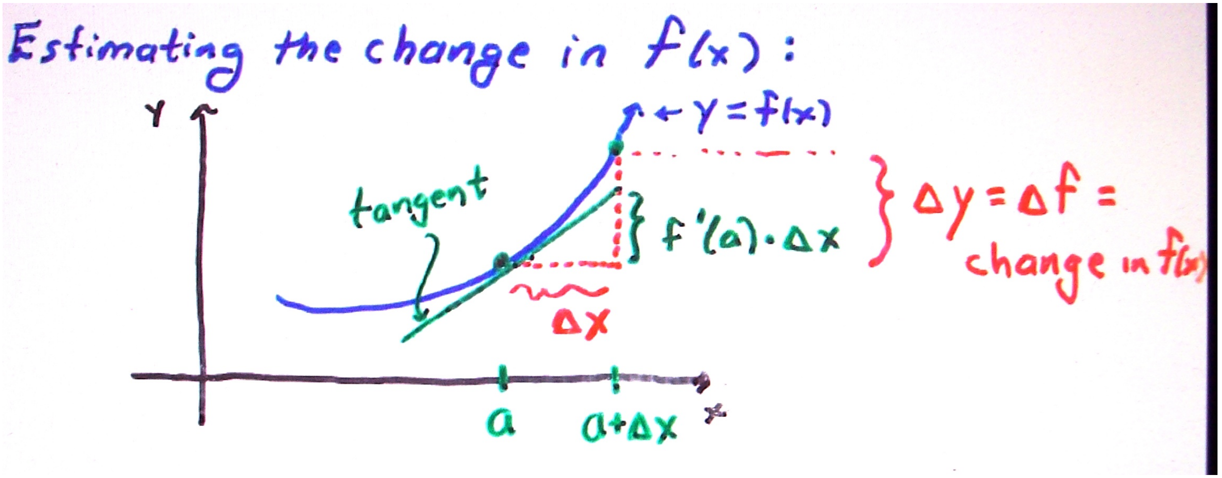
If we write $x = a + \Delta x$ and let $\Delta x$ be sufficiently small this becomes $f(a + \Delta x) - f(a) \approx f'(a)\Delta x$. Writing also $\Delta y = \Delta f := f(a + \Delta x) - f(a)$ this becomes

$$\Delta y = \Delta f \approx f'(a)\Delta x$$

In words: for small $\Delta x$ the **change** $\Delta y$ in $y$ if one goes from $x$ to $x + \Delta x$ is approximately equal to $f'(a)\Delta x$.

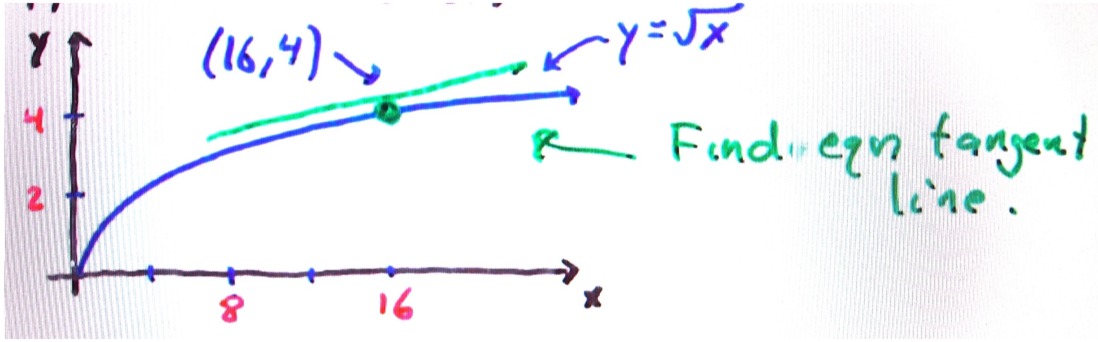
# Visualization of Linear Approximation



**Example:** a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 16$.
b) Use it to approximate $\sqrt{15.9}$.

**Solution:**




a) We have to compute the equation of the tangent line at $x = 16$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/2} = \frac{1}{2} x^{-1/2}$$

$$f'(16) = \tfrac{1}{2} 16^{-1/2} = \tfrac{1}{2} \cdot \tfrac{1}{\sqrt{16}} = \tfrac{1}{8}$$

$$L(x) = f'(a)(x - a) + f(a)$$

$$= \tfrac{1}{8}(x - 16) + \sqrt{16} = \tfrac{1}{8}x - 2 + 4 = \tfrac{1}{8}x + 2$$

b) $\sqrt{15.9} = f(15.9)$

$$\approx L(15.9) = \tfrac{1}{8} \cdot 15.9 + 2 = \tfrac{1}{8}(16 - .1) + 2 = 4 - \tfrac{1}{80} = \tfrac{319}{80}.$$

**Example:** Estimate $\cos(\frac{\pi}{4} + 0.01) - \cos(\frac{\pi}{4})$.

**Solution:**

Let $f(x) = \cos(x)$. Then we have to find $\Delta f = f(a + \Delta x) - f(a)$ for $a = \frac{\pi}{4}$ and $\Delta x = .01$ (which is small).

Using linear approximation we have:

$$\begin{aligned} \Delta f &\approx f'(a) \cdot \Delta x \\ &= -\sin(\frac{\pi}{4}) \cdot .01 \qquad \text{(since } f'(x) = -\sin x\text{)} \\ &= -\frac{\sqrt{2}}{2} \cdot \frac{1}{100} = -\frac{\sqrt{2}}{200} \end{aligned}$$

**Example:** The radius of a sphere is increased from 10 cm to 10.1 cm. Estimate the change in volume.

**Solution:**

$$V = \tfrac{4}{3}\pi r^3 \qquad \text{(volume of a sphere)}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\Delta V \approx \frac{\mathrm{d}V}{\mathrm{d}r} \cdot \Delta r = 4\pi r^2 \cdot \Delta r$$

$$= 4\pi \cdot 10^2 \cdot (10.1 - 10) = 400\pi \cdot \frac{1}{10} = 40\pi$$

The volume of the sphere is increased by $40\pi\ \mathrm{cm}^3$.

**Example:** The radius of a disk is measured to be $10 \pm .1$ cm (error estimate). Estimate the maximum error in the approximate area of the disk.

**Solution:**

$$A = \pi r^2 \qquad \text{(area of a disk)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(\pi \cdot r^2) = 2\pi r$$

$$\Delta A \approx \frac{\mathrm{d}A}{\mathrm{d}r} \cdot \Delta r = 2\pi r \cdot \Delta r$$

$$= 2\pi \cdot 10 \cdot (\pm 0.1) = \pm 20\pi \cdot \frac{1}{10} = \pm 2\pi$$

The area of the disk has approximately a maximal error of $2\pi\ \mathrm{cm}^2$.

**Example:** The dimensions of a rectangle are measured to be $10 \pm 0.1$ by $5 \pm 0.2$ inches.
What is the approximate uncertainty in the area measured?

**Solution:** We have $A = xy$ with $x = 10 \pm 0.1$ and $y = 5 \pm 0.2$.

We estimate the measurement error $\Delta_x A$ with respect to the variable $x$ and the error $\Delta_y A$ with respect to the variable $y$.

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(xy) = y$$
$$\frac{\mathrm{d}A}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}(xy) = x$$
$$\Delta_x A \approx \frac{\mathrm{d}A}{\mathrm{d}x} \cdot \Delta x = y \cdot \Delta x$$
$$\Delta_y A \approx \frac{\mathrm{d}A}{\mathrm{d}y} \cdot \Delta y = x \cdot \Delta y$$

The **total** estimated uncertainty is:

$$\Delta A = \Delta_x A + \Delta_y A = y \cdot \Delta x + x \cdot \Delta y$$
$$\Delta A = 5 \cdot (\pm .1) + 10 \cdot (\pm .2) = \pm .5 + \pm 2.0 = \pm 2.5$$

The uncertainty in the area is approximately of 2.5 $\mathrm{inches}^2$.

# Differentials

Those are a the most murky objects in Calculus I. The way they are usually defined in in calculus books is difficult to understand for a Mathematician and maybe for students, too.

Remember that we have

$$\frac{\Delta y}{\Delta x} \approx \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

The idea is to consider $\mathrm{d}y$ and $\mathrm{d}x$ as infinitesimal small numbers such that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not just an approximation but equals $f'(a)$ and one rewrites this as

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x.$$

This obviously makes no sense since the only "infinitesimal small number" which I know in calculus is 0 which gives the true but useless equation $0 = f'(x) \cdot 0$ and the nonsense equation $\frac{0}{0} = f'(x)$.

So the official explanation is that

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x$$

describes the linear approximation for the **tangent line** to $f(x)$ at the point $x$ which gives indeed this equation. Then $\mathrm{d}x$ and $\mathrm{d}y$ are numbers satisfying this equation. One problem is that one does not like to keep $x$ fixed and $f'(x)$ varies with $x$. But how to understand the dependence of $\mathrm{d}x$ and $\mathrm{d}y$ on $x$?

The symbolic explanation is that

$$\mathrm{d}y = f'(x) \cdot \mathrm{d}x$$

is an equation between the old variable $x$ and the new variables $\mathrm{d}x$ and $\mathrm{d}y$ but we **never will plug in numbers** for $\mathrm{d}x$ and $\mathrm{d}y$.

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ makes sense with both interpretations for $\mathrm{d}x \neq 0$.

For applications (substitution in integrals) we will usually need the second interpretation

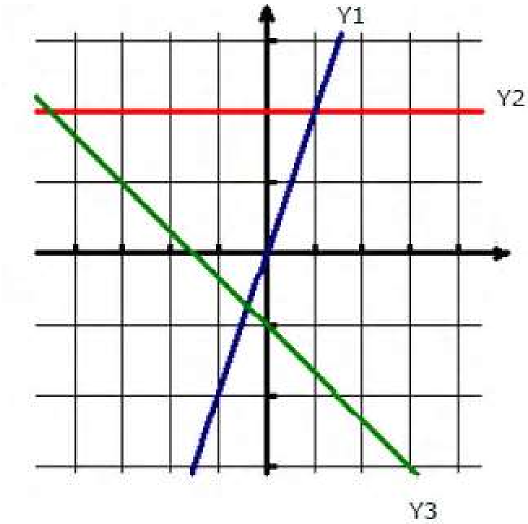
**Example:** Express $\mathrm{d}x$ in terms of $\mathrm{d}y$ for the function $y = e^{x^2}$.

**Solution:**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2} \cdot 2x$$

$$\mathrm{d}y = e^{x^2} \cdot 2x \cdot \mathrm{d}x$$

$$\mathrm{d}x = \frac{1}{e^{x^2} \cdot 2x} \cdot \mathrm{d}y = \frac{\mathrm{d}y}{2xy}$$




**Do Now:**

**1.** Write the simplest rule possible for each of the functions based solely on the graph.

$y_1 =$ 2x

$y_2 =$ 2

$y_3 =$ $-\frac{2}{3}x-1$

**2.** The above graphs are actually functions displayed on a zoomed in window. Match each graph with the correct function. Justify your answer.

$y = x^3 + 0.002$ $y_2 = 2$

$y = -\frac{2}{3}x - 0.001$ $y_3 = -\frac{2}{3}x-1$

$y = \sin(2x)$ $y_1 = 2x$

**3.** What is the scale of the above graph? .001

**4.** Algebraically find the equation of the tangent line at $x = 0$ for each function in question 2.

$y' = 3x^2$
$y'|_0 = 0$ (0, .002)
$y = .002$

$y' = -\frac{2}{3}$
$y'|_0 = -\frac{2}{3}$ (0, -.001)
$y = -\frac{2}{3}x - .001$

$y' = 2\cos(2x)$
$y'|_0 = 2$ (0, 0)
$y = 2x$

All differentiable curves are **locally linear**, since we can make the curve appear linear if we get close enough to a specific point. The *tangent line* provides a useful representation of the curve itself if we stay close enough to the point of tangency and creates the **linear approximation** at that point.

**Class Work and Homework:**

**1.** Graph the equation $y = \tan\left(\frac{x}{2}\right)$ on a zoom 4 decimal window. Zoom into a very small window.

Write the equation of the line that can be used as the linear approximation (i.e. the tangent line) for this function at $x = 0$. Graph both in the same window.

$y' = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$y'|_{x=0} = \frac{1}{2}$

$y - 0 = \frac{1}{2}(x - 0)$

$y = \frac{1}{2}x$

**2.** Find the linearization of $f(x) = \sqrt{1+x}$ (i.e. the tangent line) at $x = 0$ and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation. Is your approximation an overestimation or an underestimation?

$f'(x) = \frac{1}{2\sqrt{1+x}} = \frac{1}{2}(1+x)^{-1/2}$

$f'(0) = \frac{1}{2}$, $f(0) = 1$ $\Big\}$ $L(x) - 1 = \frac{1}{2}(x - 0)$

$L(x) = \frac{1}{2}x + 1$

$\sqrt{1.02} \approx L(.02) = \frac{1}{2}(.02) + 1 = 1.01$

$\sqrt{1.02} = f(.02) = 1.009950494$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = \frac{-1}{4(1+x)^{3/2}}$

$f''(0) = \frac{-1}{4} < 0$ concave down (tangent line above curve) $\rightarrow$

Overestimation

**3.** The slope of a function at any point $(x, y)$ is $-\frac{x+1}{y}$. The point (3, 2) is on the graph of $f$.

(a) Write an equation of the line tangent to the graph of $f$ at $x = 3$. $y - 2 = -2(x - 3)$

$y = -2x + 8$

(b) Use the tangent line in part (a) to approximate $f(3.1)$.

$y' = -\frac{x+1}{y}$

$y'|_{(3,2)} = -2$

$L(3.1) = -2(3.1) + 8 = 1.8$

$f(3.1) \approx L(3.1) = 1.8$

**4.** Let $f$ be the function that is differentiable for all real numbers. The table below gives the values of $f$ and its derivative for selected values in the interval $-0.9 \le x \le 0.9$. The second derivative is always positve, which means the function is concave up, on the same closed interval. Write an equation of the line tangent to the graph of $f$ where $x = -0.6$. Use this line to approximate the value of $f(-0.5)$. Is this approximation greater or less than the actual value of $f(-0.5)$? Give a reason to support your answer.

| $x$ | -0.9 | -0.6 | -0.3 | 0 | 0.3 | 0.6 | 0.9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | -34 | -87 | -99 | -100 | -84 | -51 | 21 |
| $f'(x)$ | -69 | -30 | -9 | 0 | 1 | 9 | 90 |

$y+87=-30(x+.6)$

$L(x)=-30x-105$

$f(-\frac{1}{2}) \approx L(-\frac{1}{2}) = -30(-\frac{1}{2})-105=-90$

$f''(x)>0 \rightarrow$ concave up (tangent line below curve)

$\therefore$ under approximation

**5.** Find the linearization of $f(x)=\cos(x)$ at $x=\frac{\pi}{2}$ and use it to approximate $\cos(1.75)$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

$f'(x)=-\sin x$

$f'(\frac{\pi}{2})=-1$

$y-0=-1(x-\frac{\pi}{2})$

$L(x)=-x+\frac{\pi}{2}$

$f(1.75) \approx L(1.75) = -1.75+\frac{\pi}{2} \approx -.179$

$f(1.75)=\cos(1.75)=-.178$

**6.** Use linearization to approximate **(a)** $\sqrt{123}$ and **(b)** $\sqrt[3]{123}$ without a calculator. Then compare the result to the actual value.

a) $f'(x)=\frac{1}{2\sqrt{x}}$

$f'(121)=\frac{1}{22}$

$y-11=\frac{1}{22}(x-121)$

$L(x)=\frac{1}{22}(x-121)+11$

$L(123)=11+\frac{1}{22}(123-121)=11+\frac{2}{22} \approx 11.\overline{09}$

$f(123)=\sqrt{123}=11.0905$

b) $f'(x)=\frac{1}{3x^{2/3}}$

$f'(125)=\frac{1}{75}$

$y-5=\frac{1}{75}(x-125)$

$L(x)=\frac{1}{75}(x-125)+5$

$L(123)=\frac{1}{75}(123-125)+5=4.97\overline{3}$

$f(123)=\sqrt[3]{123}=4.9732$

**7.** Estimate $(16.5)^{\frac{1}{4}} - 16^{\frac{1}{4}}$ using linear approximation.

$f(x) = x^{\frac{1}{4}}$

$f'(x) = \frac{1}{4x^{3/4}}$

$f'(16) = \frac{1}{32}$

$L(x) = \frac{1}{32}(x-16) + 2$

$L(16.5) = \frac{1}{64} + 2$

$(16.5)^{\frac{1}{4}} - 16^{\frac{1}{4}} \approx \frac{1}{64} + 2 - 2 = \boxed{\frac{1}{64}}$

**8.** Approximate the value of sin 31° using radians.

$f'(x) = \cos(x)$

$f'(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$L(x) = \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) + \frac{1}{2}$

$L(\frac{\pi}{6} + \frac{\pi}{180}) = \frac{1}{2} + \frac{\sqrt{3}}{2}(\frac{\pi}{6} + \frac{\pi}{180} - \frac{\pi}{6}) = \frac{1}{2} + \frac{\sqrt{3}\pi}{360} = \boxed{0.515115}$

$\sin(31) = .515038$

**9.** If $f$ is a differentiable function, $f(2) = 6$ and $f'(2) = -\frac{1}{2}$, find the approximate value of $f(2.1)$.

$L(x) = -\frac{1}{2}(x-2) + 6$

$\boxed{f(2.1) \approx L(2.1) = 5.95}$

**10.** Write an equation of the tangent line to $f(x) = x^3$ at (2, 8). Use the tangent line to find the approximate values of $f(1.9)$ and $f(2.01)$. Are these estimations greater or less than the actual value of $f(1.9)$ and $f(2.01)$?

$f'(x) = 3x^2$

$f'(2) = 12$

$L(x) = 12(x-2) + 8$

$f(1.9) \approx L(1.9) = 6.8$

$f(2.1) \approx L(2.1) = 9.2$

$f''(x) = 6x$

$f''(2) = 12 > 0$ concave up (tangent line below)

**underapproximation**

# Linear Approximation

## Introduction

By now we have seen many examples in which we determined the tangent line to the graph of a function $f(x)$ at a point $x = a$. A ***linear approximation*** (or ***tangent line approximation***) is the simple idea of using the equation of the tangent line to approximate values of $f(x)$ for $x$ near $x = a$.

A picture really tells the whole story here. Take a look at the figure below in which the graph of a function $f(x)$ is plotted along with its tangent line at $x = a$. Notice how, near the point of contact $(a, f(a))$, the tangent line nearly coincides with the graph of $f(x)$, while the distance between the tangent line and graph grows as $x$ moves away from $a$.

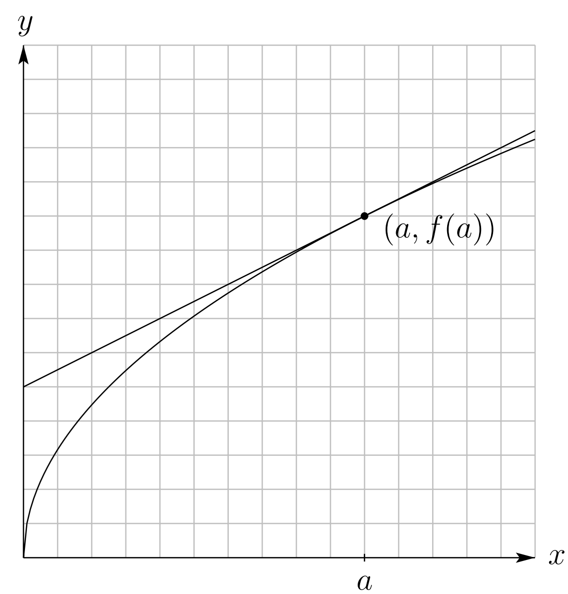



Figure 1: Graph of $f(x)$ with tangent line at $x = a$

In other words, for a given value of $x$ close to $a$, the difference between the corresponding $y$ value on the graph of $f(x)$ and the $y$ value on the tangent line is very small.

# The Linear Approximation Formula

Translating our observations about graphs into practical formulas is easy. The tangent line in Figure 1 has slope $f'(a)$ and passes through the point $(a, f(a))$, and so using the point-slope formula $y - y_0 = m(x - x_0)$, the equation of the tangent line can be expressed

$$y - f(a) = f'(a)(x - a),$$

or equivalently, isolating $y$,

$$y = f(a) + f'(a)(x - a)\ .$$

(Observe how this last equation gives us a new simple and efficient formula for the equation of the tangent line.) Again, the idea in linear approximation is to approximate the $y$ values on the graph $y = f(x)$ with the $y$ values of the tangent line $y = f(a) + f'(a)(x - a)$, so long as $x$ is not too far away from $a$. That is,

$$\boxed{\text{for } x \text{ near } a,\ f(x) \approx f(a) + f'(a)(x - a)\ .} \qquad (1)$$

Equation (1) is called the linear approximation (or tangent line approximation) of $f(x)$ at $x = a$. (Instead of "at", some books use "about", or "near", but it means the same thing.)

Notice how we use "$\approx$" instead of "$=$" to indicate that $f(x)$ is being approximated. Also notice that if we set $x = a$ in Equation (1) we get true equality, which makes sense since the graphs of $f(x)$ and the tangent line coincide at $x = a$.

# A Simple Example

Let's look at a simple example: consider the function $f(x) = \sqrt{x}$. The tangent line to $f(x)$ at $x = 1$ is $y = x/2 + 1/2$ (so here $a = 1$ is the $x$ value at which we are finding the tangent line.) This is actually the function and tangent line plotted in Figure 1. So here, for $x$ near $x = 1$,

$$\sqrt{x} \approx \frac{x}{2} + \frac{1}{2}\ .$$

To see how well the approximation works, let's approximate $\sqrt{1.1}$:

$$\begin{aligned}\sqrt{1.1} &\approx \frac{1.1}{2} + \frac{1}{2} \\ &= 1.05\end{aligned}$$

Using a calculator, we find $\sqrt{1.1} \doteq 1.0488$ to four decimal places. So our approximation has an error of about 0.1%; not bad considering the simplicity of the calculation in the linear approximation!

On the other hand, if we try to use the same linear approximation for an $x$ value far from $x = 1$, the results are not so good. For example, let's approximate $\sqrt{0.25}$:

$$\begin{aligned}\sqrt{0.25} &\approx \frac{0.25}{2} + \frac{1}{2} \\ &= 0.625\end{aligned}$$

The exact value is $\sqrt{0.25} = 0.5$, so our approximation has an error of 25%, a pretty poor approximation.

# More Examples

**Example 1:** Find the linear approximation of $f(x) = x\sin(\pi x^2)$ about $x = 2$. Use the approximation to estimate $f(1.99)$.

**Solution:** Here $a = 2$ so we need $f(2)$ and $f'(2)$:

$$f(2) = 2\sin(4\pi) = 0,$$

while

$$f'(x) = \sin(\pi x^2) + x\cos(\pi x^2)\,2\pi x\ ,$$

so that

$$f'(2) = \sin(4\pi) + 8\pi\cos(4\pi) = 8\pi\ .$$

Therefore the linear approximation is

$$f(x) \approx f(2) + f'(2)(x-2)\ ,$$

i.e.

$$\text{for } x \text{ near } 2,\ x\sin(\pi x^2) \approx 8\pi(x-2)\ .$$

Using this to estimate $f(1.99)$, we find

$$f(1.99) \approx 8\pi(1.99-2) = -0.08\pi \doteq -0.251$$

to three decimals. (Checking with a calculator we find $f(1.99) \doteq -0.248$ to three decimals.)

■

**Example 2:** Use a tangent line approximation to estimate $\sqrt[3]{28}$ to 4 decimal places.

**Solution:** In this example we must come up with the appropriate function and point at which to find the equation of the tangent line. Since we wish to estimate $\sqrt[3]{28}$, $f(x) = x^{1/3}$. For the $a$-value in Equation (1) we ask: at what value of $x$ near 28 do we know $f(x)$ exactly? Answer: $x = 27$, which is a perfect cube.

Thus, using $f(x) = x^{1/3}$ we find $f'(x) = (1/3)x^{-2/3}$, so that $f(27) = 3$ and $f'(27) = 1/27$. The linear approximation formula is then

$$f(x) \approx f(27) + f'(27)(x - 27) \ ,$$

i.e., for $x$ near 27,

$$x^{1/3} \approx 3 + \frac{1}{27}(x - 27) \ .$$

Using this to approximate $\sqrt[3]{28}$ we find

$$\begin{aligned} \sqrt[3]{28} &\approx 3 + \frac{1}{27}(28 - 27) \\ &= \frac{82}{27} \\ &\doteq 3.0370 \end{aligned}$$

A calculator check gives $\sqrt[3]{28} \doteq 3.0366$ to 4 decimals.

■

**Example 3:** Consider the implicit function defined by

$$3(x^2 + y^2)^2 = 100xy \ .$$

Use a tangent line approximation at the point $(3, 1)$ to estimate the value of $y$ when $x = 3.1$.

**Solution:** Even though $y$ is defined implicitly as a function of $x$ here, the tangent line to the graph of $3(x^2 + y^2)^2 = 100xy$ at $(3, 1)$ can easily be found and used to estimate $y$ for $x$ near 3.

First, find $y'$. Differentiating both sides of $3(x^2 + y^2)^2 = 100xy$ with respect to $x$ gives

$$6(x^2 + y^2)(2x + 2yy') = 100y + 100xy' \ .$$

Now substitute $(x, y) = (3, 1)$:

$$6(9 + 1)(6 + 2y') = 100 + 300y'$$

which yields $y' = 13/9$. Thus the equation of the tangent line is

$$\begin{aligned} y - 1 &= \frac{13}{9}(x - 3), \ \text{or} \\ y &= \frac{13}{9}x - \frac{30}{9} \ . \end{aligned}$$

Thus, for points $(x, y)$ on the graph of $3(x^2 + y^2)^2 = 100xy$ with $x$ near 3,

$$y \approx \frac{13}{9}x - \frac{30}{9} \ .$$

Setting $x = 3.1$ in this last equation gives $y \approx 103/90 \doteq 1.14$ to two decimals.

■

# Exercises

1. Physicists often use the approximation $\sin x \approx x$ for small $x$. Convince yourself that this is valid by finding the linear approximation of $f(x) = \sin x$ at $x = 0$.

   **Solution** For $x$ near 0, $f(x) \approx f(0) + f'(0)(x - 0)$. Using $f(x) = \sin x$, $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$ we find $\sin x \approx x$.

2. Find the linear approximation of $f(x) = x^3 - x$ about $x = 1$ and use it to estimate $f(0.9)$.

   **Solution** For $x$ near 1, $f(x) \approx f(1) + f'(1)(x - 1)$. Using $f(x) = x^3 - x$, $f(1) = 0$ and $f'(1) = 2$ we find $f(x) \approx 2(x - 1)$, so $f(0.9) \approx 2(0.9 - 1) = -0.2$.

3. Use a linear approximation to estimate $\cos 62°$ to three decimal places. Check your estimate using your calculator. For this problem recall the trig value of the special angles:

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $\pi/4$ | $1/\sqrt{2}$ | $1/\sqrt{2}$ | $1$ |
| $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $1/\sqrt{3}$ |

   **Solution** Here $62°$ is close to $60°$ which is $\pi/3$ radians, and we know $\cos(\pi/3) = 1/2$. Letting $f(x) = \cos x$, for $x$ near $\pi/3$, $f(x) \approx f(\pi/3) + f'(\pi/3)(x - \pi/3)$. Since $62° = 62\pi/180$ radians and $f'(x) = -\sin x$, this gives

$$\begin{aligned}\cos 62° &\approx 1/2 - \sin(\pi/3)(62\pi/180 - \pi/3)\\ &= 1/2 - (\sqrt{3}/2)(\pi/90)\\ &\doteq 0.470\end{aligned}$$

4. Use a tangent line approximation to estimate $\sqrt[4]{15}$ to three decimal places.

   **Solution** 15 is near 16 where we know $\sqrt[4]{16} = 2$ exactly. Letting $f(x) = \sqrt[4]{x}$, we have for $x$ near 16, $f(x) \approx f(16) + f'(16)(x - 16)$. That is, $\sqrt[4]{x} \approx 2 + (1/32)(x - 16)$. Thus

$$\begin{aligned}\sqrt[4]{x} &\approx 2 + (1/32)(15 - 16)\\ &= 63/32\\ &\doteq 1.969 \ .\end{aligned}$$

5. Define $y$ implicitly as a function of $x$ via $x^{2/3} + y^{2/3} = 5$. Use a tangent line approximation at $(8, 1)$ to estimate the value of $y$ when $x = 7$.

   **Solution** First find the equation of the tangent line to the curve at $(8, 1)$ and then substitute $x = 7$. Differentiating implicity with respect to $x$ we find

   $$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

   and substituting $(x, y) = (8, 1)$ yields $y' = -1/2$. Thus the equation of the tangent line is

   $$y = 1 - \frac{1}{2}(x - 8)$$

   and substituting $x = 7$ we find $y = 3/2$. That is, $(7, 3/2)$ is the point on the tangent line. Thus the point on the curve with $x$ coordinate $x = 7$ has corresponding $y$ coordinate $y \approx 3/2$.

6. Suppose $f(x)$ is a differentiable function whose graph passes through the points $(-1, 4)$ and $(1, 7)$. The estimate $f(-0.8) \approx 5$ is obtained using a linear approximation about $x = -1$. Using this information, find $\left.\frac{d}{dx}(f(x))^2\right|_{x=-1}$.

   **Solution** This problem was made more difficult by adding extra information which is not needed for the solution: the point $(1, 7)$ plays no part. First, note that since $(-1, 4)$ is on the graph of $f(x)$, $f(-1) = 4$. For $x$ near $-1$, $f(x) \approx f(-1) + f'(-1)(x + 1)$. Using this linear approximation, the estimate $f(-0.8) \approx 5$ was made; that is

   $$5 = 4 + f'(-1)(-0.8 + 1)$$

   So that $f'(-1) = 5$. Now do the derivative, remembering the chain rule:

   $$\begin{aligned}\left.\frac{d}{dx}(f(x))^2\right|_{x=-1} &= \left.2f(x)f'(x)\right|_{x=-1} \\ &= 2(4)(5) \\ &= 40\ .\end{aligned}$$

7. The profit $P(q)$ from producing $q$ units of goods is given by

   $$P(q) = 396q - 2.2q^2 + k$$

   for some constant $k$. Using a linear approximation about $q = 80$ we find $P(81) \approx 17244$. What is $k$?

   **Solution** For $q$ near 80, $P(q) \approx P(80) + P'(80)(q - 80)$. Using this approximation, $P(81) \approx 17244$, so that

   $$\begin{aligned}17244 &= P(80) + P'(80)(q - 80) \\ 17244 &= [396(80) - 2.2(80)^2 + k] + [396 - 4.4(80)](1)\end{aligned}$$

   where in this last equation the first expression in square brackets is $P(80)$ and the second expression in square brackets is $P'(80)$. Solving this last equation for $k$ gives $k = -400$ (note the original answers had $k = 400$ which is incorrect).