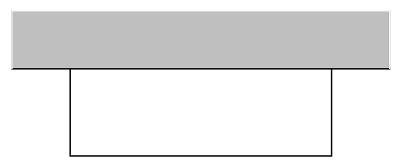
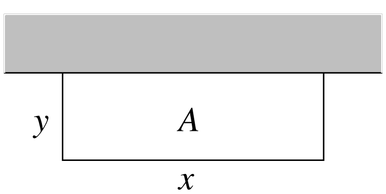
**Example:** Suppose you have 30 ft of fencing and want to fence in a rectangular garden next to a house. What should the dimensions of the rectangle be so that the garden is as large as possible?

| Procedure | Solution |
|---|---|
| **Read the problem, then read it again**. Begin by reading the problem. It's okay if not everything makes sense at first, but try and get an idea of what the problem about and what it's asking you to do. What types of quantities are involved? What are their units? | Problem will involve distance (in feet) and area (units are sq. ft) |
| **Draw a picture**. In most cases you should start with a sketch. Don't worry about artistic quality; the point of the diagram is to help you visualize the problem, provide a place for you to label unknown quantities, and ultimately help you translate the problem into mathematics. | Here is a top-down view of the fenced garden. The shaded region represents the solid wall of the house. The rectangle next to it is the garden with the fence along the three sides that are not adjacent to the house.<br> |
| **Name and label unknown quantities**. Make up appropriate variable names for quantities you don't know and think you might need. Put these on your diagram. | Let $x$ be the width and $y$ be the height of the rectangle and use $A$ for the area of the garden. The names used here are somewhat arbitrary. We could have just as easily used $l$ and $w$ for length and width.<br><br> |
| **Determine what you're trying to optimize**. Clearly identify the *objective*, which is the quantity you want to maximize or minimize. | In this problem we want to maximize the area, so our objective is $A$. |
| **Develop an equation with the objective**. Develop an equation that relates the objective to other known and unknown (but named) quantities. Use your diagram and pay close attention to units. | Area is length times width: $A = xy$. |

| Procedure | Solution |
|---|---|
| **Write the objective in terms of a single variable**. Use algebra and additional relationships to rewrite the objective equation so it expresses the objective as a function of a single variable. | Our objective has two variables, $x$ and $y$ and we want to eliminate one of them by expressing it in terms of the other. Recall that we were told the length of the fence was 30 ft, so $30 = x + 2y$. Solving for $x$ we have $x = 30 - 2y$, so we can replace $x$ in our objective with $30 - 2y$. $$A = (30 - 2y)y = 30y - 2y^2$$ |
| **Determine the domain**. Clearly identify the set of values the independent variable can take on. The endpoints of this domain (if they exist) may correspond to extrema. | We note that area cannot be negative, so $A \geq 0$. Thus $(30 - 2y)y \geq 0$. This means $0 \leq y \leq 15$. The domain of our objective is the closed interval $[0, 15]$. |
| **Find all extrema.** Finally we get to use calculus! Usually this means to find the absolute extreme value of the objective. Remember, this means we need to identify and check all critical numbers as well as the endpoints of our domain. | Computing the derivative of $A$ with respect to $y$ we have $A' = 30 - 4y$. Since this is defined for all $y$ we know that the only critical value occurs where $30 - 4y = 0$, which gives $y = 7.5$. We need to check three values for our absolute maximum: left endpoint $y = 0$, critical value $y = 7.5$, and right endpoint $y = 15$: $$A(0) = 0, \quad A(7.5) = 112.5, \quad A(15) = 0$$ So our absolute maximum is 112.5 and it occurs when $y = 7.5$. |
| **Review the problem statement**. Pay special attention to the form of the question or problem. Does your answer make sense? Are the units correct? | Our problem statement asked for the dimensions of the rectangle that provides the largest garden. We know one dimension, 7.5 ft, but still the other. Since $x = 30 - 2y$ we easily compute $x = 30 - 2 \cdot 7.5 = 15$ ft. |
| **State your answer**. Okay, you've done the math, now you need to communicate your answer clearly. Provide an answer for the problem using a **complete sentence**. *Note: The variable names you introduced when solving your problem should not appear in your answer!* | **Answer:** To maximize the garden area, the rectangle should be $7.5' \times 15'$, with the long side parallel to the house wall. |

**Note:** Not all of the steps will be required to solve every optimization problem.

# Calculus Optimization Problems/Related Rates Problems Solutions

1) A farmer has 400 yards of fencing and wishes to fence three sides of a rectangular field (the fourth side is along an existing stone wall, and needs no additional fencing). Find the dimensions of the rectangular field of largest area that can be fenced.

$2x + y = 400 \Rightarrow y = 400 - 2x$

$A(x) = x(400 - 2x) = 400x - 2x^2$

$A'(x) = 400 - 4x \quad 400 - 4x = 0 \Rightarrow x = 100$

$A''(x) = -4$

By the 2nd derivative test, the dimensions would be 100 yd by 200 yd.

2) A metal box (without a top) is to be constructed from a square sheet of metal that is 20 cm on a side by cutting square pieces of the same size from the corners of the sheet and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed in this manner.

$V(x) = x(20 - 2x)(20 - 2x) = 400x - 80x^2 + 4x^3$

$V'(x) = 400 - 160x + 12x^2$

$400 - 160x + 12x^2 = 0 \Rightarrow 4(100 - 40x + 3x^2) = 0 \Rightarrow 4(3x - 10)(x - 10) \Rightarrow x = \frac{10}{3}, 10$

$V''(x) = -160 + 24x \quad V''\left(\frac{10}{3}\right) = -160 + 80 < 0 \quad V''(10) = -160 + 240 > 0$

By the 2nd derivative test, the dimensions would be $\frac{10}{3}$ cm by $\frac{40}{3}$ cm by $\frac{40}{3}$ cm

3) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$5 per meter, and the fencing for the other sides costs \$3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose.

Call the length of fence along the river $x$, and the length perpendicular to the river $y$.

$C(x) = 5x + 3(2y + x) \qquad xy = 1200 \Rightarrow y = \frac{1200}{x} \Rightarrow C(x) = 8x + \frac{7200}{x}$

$C'(x) = 8 - \frac{7200}{x^2} \quad 8 - \frac{7200}{x^2} = 0 \Rightarrow 8x^2 = 7200 \Rightarrow x^2 = 900 \Rightarrow x = 30$

$C''(x) = \frac{14400}{x^3} \quad C''(30) = \frac{14400}{30^3} > 0$

By the 2nd derivative test, a field that is 30 m along the river by 40 m perpendicular to the river would be least expensive.

4) A 4-meter length of stiff wire is cut in two pieces. One piece is bent into the shape of a square and the other into a rectangle whose length is 3 times its width. Let $x$ be the length of the side of the square.
a) Find a formula $A(x)$, the sum of the areas of the square and rectangle, in terms of the variable $x$.

The length of wire left for the rectangle is $4 - 4x$. In the rectangle, $l = 3w$. $4 - 4x = 2(3w) + 2w$, so $w = \frac{4 - 4x}{8} = \frac{1 - x}{2} \Rightarrow l = \frac{3 - 3x}{2}$. $A(x) = x^2 + \left(\frac{3 - 3x}{2}\right)\left(\frac{1 - x}{2}\right)$

b) For what values of $x$ does $A(x)$ achieve its maximum; for which does it achieve its minimum. Justify your answer.

$$x^2 + \left(\frac{3-3x}{2}\right)\left(\frac{1-x}{2}\right) = 0 \Rightarrow x^2 + \frac{3-6x+3x^2}{4} = 0 \Rightarrow 7x^2 - 6x + 3 = 0$$

5) A rectangular playing field is to have area 600 m$^2$. Fencing is required to enclose the field and to divide it into two equal halves.

a) Find a formula, $F(x)$, for the total length of fencing required, in terms of the length, $x$, of the fence dividing the field in half.

$$F(x) = 3x + 2\left(\frac{600}{x}\right) = 3x + \frac{1200}{x}$$

b) Find the minimum amount of fencing needed to do this.

$$F'(x) = 3 - \frac{1200}{x^2} \quad 3 - \frac{1200}{x^2} = 0 \Rightarrow 3x^2 = 1200 \Rightarrow x = 20$$

$$F''(x) = \frac{2400}{x^3} \quad F''(20) = \frac{2400}{20^3} > 0$$

By the 2nd derivative test, The minimum amount of fencing needed is 120 m

c) What are the outer dimensions of the field that has the least fencing? 20 m by 30 m

6) A rectangle has its base on the $x$-axis and its upper vertices on the parabola $y = 27 - x^2$. Find the maximum possible area of the rectangle.

$$A(x) = 2x\left(27 - x^2\right) = 54x - 2x^3$$

$$A'(x) = 54 - 6x^2 \quad 54 - 6x^2 = 0 \Rightarrow x^2 = 9 \Rightarrow x = 3$$

$$A''(x) = -12x \quad A''(3) = -36 < 0$$

By the 2nd derivative test, the maximum area would be 6(18) = 108 sq units.

7) A rectangular container with open top is required to have a volume of 16 cubic meters. Also, one side of the rectangular base is required to be 4 meters long. If material for the base costs \$8 per square meter, and material for the sides costs \$2 per square meter, find the dimensions of the container so that the cost of material to make it will be a minimum.

$$V = 4wh = 16 \Rightarrow h = \frac{4}{w}$$

$$C = 8(4w) + 2(2wh) + 2(2(4h)) = 32w + 16 + \frac{64}{w}$$

$$C' = 32 - \frac{64}{w^2} \quad 32 - \frac{64}{w^2} = 0 \Rightarrow w = \sqrt{2}$$

$$C'' = \frac{128}{w^3} \quad \frac{128}{\sqrt{2}^3} > 0$$

By the 2nd derivative test, the dimensions of the container that minimizes the cost are 4 m by $\sqrt{2}$ m (base) by $\frac{4}{\sqrt{2}}$ m (height)

8) A rectangular box with open top is to be constructed from a rectangular piece of cardboard 80 cm by 30 cm, by cutting out equal squares from each corner of the sheet of cardboard and folding up the resulting flaps. Find the dimensions of the box of maximum volume made by these conditions.

$$V = x(80-2x)(30-2x) = 2400x - 220x^2 + 4x^3$$

$$V' = 2400 - 440x + 12x^2 = 4(3x^2 - 110x + 600)$$

$$4(3x^2 - 110x + 600) = 0 \Rightarrow 4(3x-20)(x-30) = 0 \Rightarrow x = \frac{20}{3}, 30$$

$$V''(x) = -440 + 24x \quad V''\left(\frac{20}{3}\right) = -440 + 160 < 0$$

By the 2nd derivative test, the dimension of the box of maximum volume are $\frac{20}{3}$ cm by $\frac{200}{3}$ cm by $\frac{50}{3}$ cm

9) Find the points on the parabola $2x + y^2 = 0$ closest to the point (-3,0).

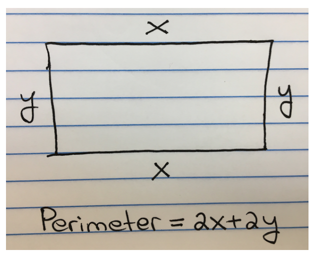
10) A power line is needed to connect a power station on the shore of a river to an island 4 miles downstream and 1 mile offshore. Find the minimum cost for such a line given that is costs \$50,000 per mile to lay wire under the water and \$30,000 per mile to lay wire underground.

## - OPTIMIZATION PROBLEMS

Steps for Solving Optimization Problems:
1) Read the problem carefully. What quantities are given to us, and which quantity needs to be optimized?
2) Draw a picture of the problem and label the relevant information.
3) Introduce variables and write down the relation between them (for example, if $A$ is the variable representing area, $L$ and $W$ are length and width, the relation would be $A = L \cdot W$).
4) Express the quantity that you want to optimize as a **single variable** function.
5) Find the critical points of this function and identify the points where the function is maximized or minimized.

**Problem 1.** Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

**Solution.** Here is the setup:




The perimeter is $2x + 2y = 100$. The function we want to maximize is the area $A = xy$. Solving for $y$, we get $y = \frac{100 - 2x}{2} = 50 - x$. So the area can be written as a function of $x$, namely $A(x) = xy = x(50 - x)$. The domain of this function is $0 < x < 50$. We have $A(x) = 50x - x^2$ so $A'(x) = 50 - 2x$. Setting $A'(x) = 0$ we get $x = 25$ as the only critical point. Note that $A''(x) = -2 < 0$ so by the second derivative test $x = 25$ is a local maximum. It is also the global maximum because as you approach the endpoints the area decreases. Thus, $x = 25$ and $y = 50 - x = 50 - 25 = 25$ are the dimensions that maximize the area. So, among the rectangles with fixed perimeter, a square is the one that maximizes the area!

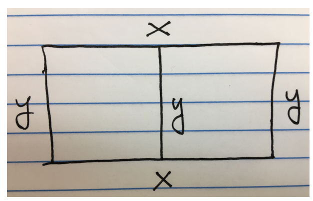
**Problem 2.** A cylindrical can is to be made to hold 1000 $\text{cm}^3$ of oil. Find the dimensions of the can that will minimize the cost of the metal when manufacturing the can.

**Solution.** The volume is $V = \pi r^2 h = 1000$ and we want to minimize the total area $A = 2\pi rh + 2\pi r^2$. We can solve $h$ in terms of $r$ by using $\pi r^2 h = 1000$. We get $h = \frac{1000}{\pi r^2}$. So $A(r) = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$ is a function of $r$ only. The domain of this function is $(0, \infty)$. So we get $A'(r) = -\frac{2000}{r^2} + 4\pi r$.

Setting $A'(r) = 0$ we get $r = \sqrt[3]{\frac{500}{\pi}}$. Next, $A''(r) = \frac{4000}{r^3} + 4\pi$ so $A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0$ so $r = \sqrt[3]{\frac{500}{\pi}}$ is a local minimum by the second derivative test. It is also global minimum, because as you approach the endpoints the surface area incrases. So the dimensions of the can that minimize the cost of the metal is $r = \sqrt[3]{\frac{500}{\pi}}$ and $h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \sqrt[3]{\left(\frac{500}{\pi}\right)^2}} = \frac{1000}{\sqrt[3]{250000\pi}} = \frac{100}{\sqrt[3]{250\pi}}$.

**Problem 3.** A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

**Solution.** Here is the setup:




The area is $xy = 1.5 \cdot 10^6$ square feet. We want to minimize the perimeter, which is $2x + 3y$. We can express $x$ in terms of $y$ via $xy = 1.5 \cdot 10^6$, that is, $x = \frac{1.5 \cdot 10^6}{y}$. Then the perimeter $P = 2x + 3y = \frac{3 \cdot 10^6}{y} + 3y$ is expressed in terms of $y$ only. So $P(y) = \frac{3 \cdot 10^6}{y} + 3y$ is the function we want to minimize. The domain of this function is $(0, \infty)$. We have

$$P'(y) = -\frac{3 \cdot 10^6}{y^2} + 3 \quad \text{Set } P'(y) = 0 \quad \Rightarrow \quad 3 = \frac{3 \cdot 10^6}{y^2} \quad \Rightarrow \quad y = 1000$$

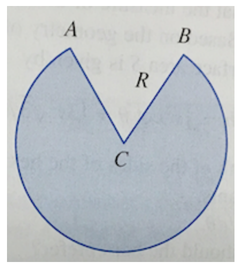
Now $P''(y) = \frac{6 \cdot 10^6}{y^3}$ so $P''(1000) > 0$ which means that $y = 1000$ is a local min (by the second derivative test). It is also a global min (as you approach the endpoints the perimeter increases). So $y = 1000$ and $x = 1.5y = 1500$ are the right dimensions of the fence in order to minimize the cost of the fence.

**Problem 4.** Find the point on the line $y = 4x + 7$ that is closest to the origin.

**Solution.** The distance between the point $(x, y)$ and the origin is $\sqrt{x^2 + y^2}$. Since we want the point $(x, y)$ to be on the line $y = 4x + 7$, the distance between the point on this particular line and the origin is $d(x) = \sqrt{x^2 + (4x + 7)^2}$. So we want to maximize the function $d(x)$. It is easier to maximize the square of this function (this is okay because the function $x \mapsto x^2$ is an increasing function, so a function and its square will achieve the maximum at the same place). The square of the distance function is $f(x) = x^2 + (4x + 7)^2$.

We have $f'(x) = 2x+2(4x+7)\cdot 4 = 2x+8(4x+7) = 34x+56$. If $f'(x) = 0$ then $34x = -56$ so $x = -56/34 = -28/17$. Note that $f''(x) = 34 > 0$ so by the second derivative test, the value $x = -28/17$ is a local min, but it is also a global min (from the geometry of the problem). For this value of $x$, the $y$-value is $y = 4x + 7 = -112/17 + 7 = -112/17 + 119/17 = 7/17$. So the coordinates of the point on the line $y = 4x + 7$ closest to the origin is $(-28/17, 7/17)$.

**Problem 5.** A cone-shaped drinking cup is made from a circular piece of paper of radius $R$ by cutting out a sector and joining the edges $CA$ and $CB$. Find the maximum capacity of such a cup.



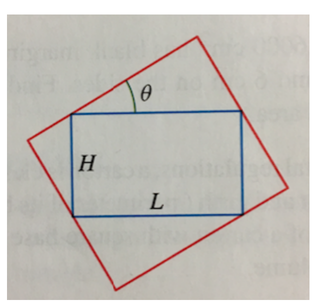

**Solution.** First, note that $R$ is a constant. After you glue the edges $CA$ and $CB$ together and make a cone, this cone will have a height $h$ and radius $r$ satisfying a constraint $r^2 + h^2 = R^2$. The volume to be maximized is $V = \frac{1}{3}\pi r^2 h$. Substituting $r^2 = R^2 - h^2$, we get

$$V(h) = \frac{1}{3}\pi(R^2 - h^2)h = \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi h^3$$

as a function of $h$ only. The domain of this function is $0 < h < R$. Now, $V'(h) = \frac{1}{3}\pi R^2 - \pi h^2$. Setting $V'(h) = 0$ we get $h^2 = \frac{1}{3}R^2$ so $h = \sqrt{\frac{1}{3}}R$ is the only critical point. Since $V''(h) = -2\pi h$ we have $V''(\sqrt{\frac{1}{3}}R) < 0$ so it is a local max by the second derivative test. It is also a global maximum. So the maximum volume is

$$\frac{1}{3}\pi\left(R^2 - \frac{1}{3}R^2\right)\sqrt{\frac{1}{3}}R = \frac{1}{3}\pi\frac{2}{3}R^2\sqrt{\frac{1}{3}}R = \frac{2\pi R^3}{9\sqrt{3}}$$

**Problem 6.** Find the maximum area of a rectangle circumscribed around a rectangle of sides $L = 5$ and $H = 3$.



**Solution.** The choice of the angle $\theta$ determines the rectangle. So our goal should be to express the area of the outer rectangle using only $\theta$. The sides of the outer rectangle is $L\sin(\theta) + H\cos(\theta)$ and $H\sin(\theta) + L\cos(\theta)$. In this case, $L = 5$ and $H = 3$, so the sides are $5\sin(\theta) + 3\cos(\theta)$ and $3\sin(\theta) + 5\cos(\theta)$. As a result, the area of the big rectangle is

$$A(\theta) = (5\sin(\theta) + 3\cos(\theta)) \cdot (3\sin(\theta) + 5\cos(\theta))$$

as a function of $\theta$. The domain of this function is $(0, \pi/2)$. Let's simplify the function:

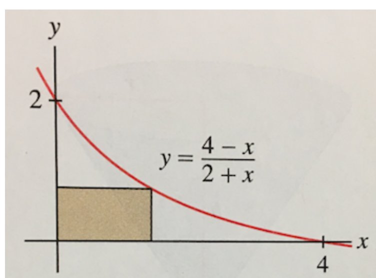
$$\begin{aligned} A(\theta) &= 15\sin^2(\theta) + 25\sin(\theta)\cos(\theta) + 9\cos(\theta)\sin(\theta) + 15\cos^2(\theta) \\ &= 15 + 34\sin(\theta)\cos(\theta) = 15 + 17\sin(2\theta) \end{aligned}$$

Now $A'(\theta) = 17\cos(2\theta) \cdot 2 = 34\cos(2\theta)$. Setting $A'(\theta) = 0$ we get $\cos(2\theta) = 0$ so $2\theta = \pi/2$, that is, $\theta = \pi/4$. Since $A''(\theta) = -67\sin(2\theta)$ and $A''(\pi/4) = -67 < 0$, this is a local max, and in fact a global maximum by the geometry of the problem. Plugging $\theta = \pi/4$ we get

$$A(\pi/4) = 15 + 17\sin(2\pi/4) = 15 + 17 = 32$$

gives the maximum area of the rectangle circumscribed around a rectangle of sides $L = 5$ and $H = 3$.

**Problem 7.** Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = \dfrac{4-x}{2+x}$.




**Solution.** Call the base of the rectangle $x$. Then the height of the rectangle is exactly $y = \dfrac{4-x}{2+x}$, which means that the area is

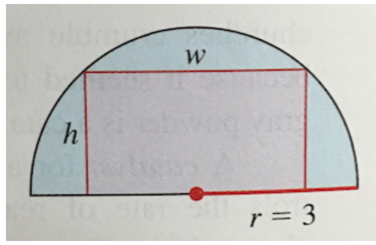
$$A = xy = x \cdot \left(\frac{4-x}{2+x}\right) = \frac{4x - x^2}{2+x}$$

This expresses area as a function of $x$ only. The domain is $(0, \infty)$. We can now try maximizing the area:

$$A'(x) = \frac{(4-2x)(2+x) - (4x - x^2)}{(2+x)^2} = \frac{8 - 2x^2 - 4x + x^2}{(2+x)^2} = \frac{-x^2 - 4x + 8}{(2+x)^2}$$

We set $A'(x) = 0$ and we get $x^2 + 4x - 8 = 0$ which gives $x = \frac{-4+\sqrt{48}}{2} = \frac{-4+4\sqrt{3}}{2} = 2\sqrt{3} - 2$. So the maximum area is:

$$(2\sqrt{3} - 2)\frac{4 - (2\sqrt{3} - 2)}{2 + (2\sqrt{3} - 2)} = (2\sqrt{3} - 2)\frac{6 - 2\sqrt{3}}{2\sqrt{3}} = 2(\sqrt{3} - 1)(\sqrt{3} - 1) = 8 - 4\sqrt{3} \approx 1.0718$$




**Problem 8.** Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3.

**Solution.** Draw the line from the dot (this is the center of the imaginary circle if we were to extend the semicircle into the full circle) to the corner of the rectangle. This is simply the radius, so it has length 3. The Pythagorean theorem tells us htat

$$h^2 + \left(\frac{w}{2}\right)^2 = 3^2 = 9$$

The area that needs to be maximized is $A = hw$. Solving for $w$ in terms of $h$, we get

$$w = 2\sqrt{9 - h^2}$$

So we want to maximize the function $A(h) = 2h\sqrt{9-h^2}$, whose domain is $(0, 3)$. We have

$$A'(h) = 2\sqrt{9-h^2} + 2h\frac{1}{2\sqrt{9-h^2}} \cdot (-2h) = 2\sqrt{9-h^2} - \frac{2h^2}{\sqrt{9-h^2}}$$

Setting $A'(h) = 0$ we obtain

$$2\sqrt{9-h^2} = \frac{2h^2}{\sqrt{9-h^2}} \;\Rightarrow\; 9 - h^2 = 2h^2 \;\Rightarrow\; 3h^2 = 9 \;\Rightarrow\; h = \sqrt{3}$$

Check that this is a local max, and in fact a global max. So the dimensions are $h = \sqrt{3}$ and $w = 2\sqrt{9-3} = 2\sqrt{6}$.

**Source.** Problems 1, 2, 3, 4 and 5 are taken from Stewart's *Calculus*, Problem 6 and 7 from Rogawski's *Calculus* and Problem 8 from Thomas' *Calculus*.

CALCULUS
WORKSHEET ON OPTIMIZATION

Work the following on notebook paper. Write a function for each problem, and justify your answers. Give all decimal answers correct to three decimal places.

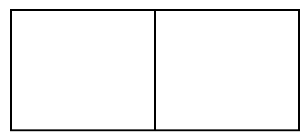
1. Find two positive numbers such that their product is 192 and the sum of the first plus three times the second is a minimum.

2. Find two positive numbers such that the sum of the first and twice the second is 100 and their product is a maximum.

3. A gardener wants to make a rectangular enclosure using a wall as one side and 120 m of fencing for the other three sides. Express the area in terms of $x$, and find the value of $x$ that gives the greatest area.

4. A rectangle has a perimeter of 80 cm. If its width is $x$, express its length and area in terms of $x$, and find the maximum area.

5. Suppose you had 102 m of fencing to make two side-by-side enclosures as shown. What is the maximum area that you could enclose?



6. Suppose you had to use exactly 200 m of fencing to make either one square enclosure or two separate square enclosures of any size you wished. What plan would give you the least area? What plan would give you the greatest area?

7. A piece of wire 40 cm long is to be cut into two pieces. One piece will be bent to form a circle; the other will be bent to form a square.

(a) Find the lengths of the two pieces that cause the sum of the area of the circle and the area of the square to be a minimum.

(b) How could you make the total area of the circle and the square a maximum?

8. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

9. The combined perimeter of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area.

10. The combined perimeter of a circle and a square is 16. Find the dimensions of the circle and square that produce a minimum total area.

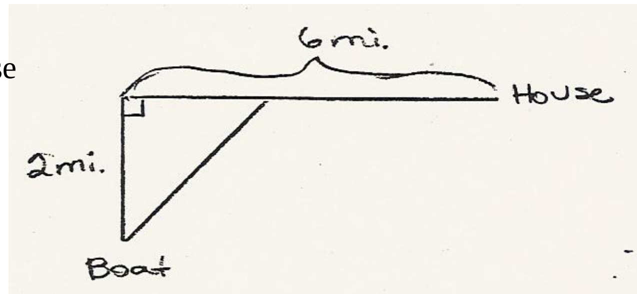
11. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

12. A rectangular page is to contain 24 sq. in. of print. The margins at the top and bottom of the page are each $1\frac{1}{2}$ inches. The margins on each side are 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

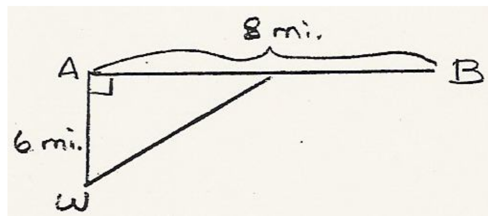
13. A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10/sq. m. for the base and $5/sq. m. for the sides, what is the cost of the least expensive tank, and what are its dimensions?

14. A cylindrical metal container, open at the top, is to have a capacity of $24\pi$ cu. in. The cost of material used for the bottom of the container is $0.15/sq. in., and the cost of the material used for the curved part is $0.05/sq. in. Find the dimensions that will minimize the cost of the material, and find the minimum cost.

15. A person in a rowboat two miles from the nearest point on a straight shoreline wishes to reach a house six miles farther down the shore. If the person can row at a rate of 3 mi/h and walk at a rate of 5 mi/h, find the least amount of time required to reach the house. How far from the house should the person land the rowboat?




16. An offshore well is located in the ocean at a point W which is six miles from the closest shore point A on a straight shoreline. The oil is to be piped to a shore point B that is eight miles from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is $100,000 per mile under water and $75,000 per mile over land, how far from A should the point P be located to minimize the cost of laying the pipe? What will the cost be?

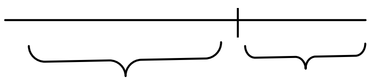


Worksheet on Optimization

1. 24 and 8
2. 50 and 25
3. Area = $x(120-2x)$
   $x$ = 30 ft.
4. Length = $40-x$
   Area = $x(40-x)$
   400 sq. ft.
5. 433.5 sq. m
6. Two squares give 1250 sq. m.
   One square gives 2500 sq. m.
7. (a) Circumference = 17.596 cm and
   perimeter of square = 22.404 cm
   (b) Just a circle with circum.0 of 40 cm
   gives area of 127.324 sq. cm.
8. All 4 ft for the circle; none for the square
9. Sides of triangle = 1.883 m and
   sides of square = 1.087 m
10. Radius of circle = 1.120 and
    sides of square = 2.240
11. 6 in. x 6 in. x 3 in.
12. 9 in x 6 in.
13. \$330, 3 x 3 x 4m
14. $r$ = 2in, $h$ = 6in, \$5.65
15. 1.733 hr, 4.5 miles
16. \$996,862.70, 6.803mi

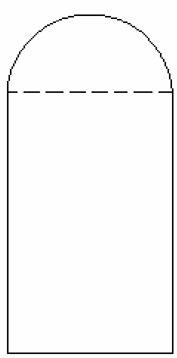
OPTIMIZATION – PART 1 NAME________________________________

1. A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a minimum? A maximum?

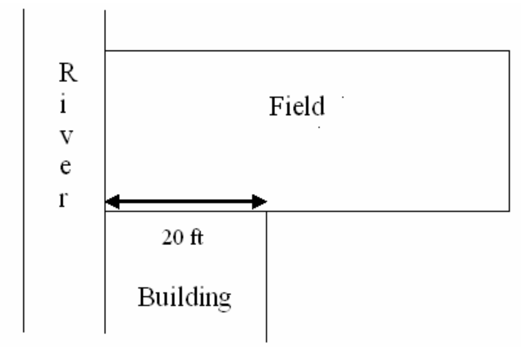


circle square

2. A window consisting of a rectangle topped by a semicircle is to have an outer perimeter *P*. Find the radius of the semicircle if the area of the window is to be a maximum.



3. A rectangular field as shown is to be bounded by a fence. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fencing. You can assume that fencing is not needed along the river and building.




4. A company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs \$10 per square foot and the rest is made with material that costs \$8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.

OPTIMIZATION – PART 1 NAME Key

1. A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a minimum? A maximum?

$A_c = \pi r^2 \qquad A_s = s^2$

$A_T = \pi r^2 + s^2$

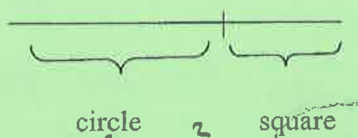
$C = 2\pi r$

$P = 4s$

$12 = 2\pi r + 4s$

$s = \dfrac{12 - 2\pi r}{4}$

$s = 3 - \frac{1}{2}\pi r$




$A_T = \pi r^2 + s^2$

$A_T = \pi r^2 + \left(3 - \frac{1}{2}\pi r\right)^2$

$\dfrac{dA}{dr} = 2\pi r + 2\left(3 - \frac{1}{2}\pi r\right)\left(-\frac{\pi}{2}\right)$

$0 = 2\pi r - 3\pi + \dfrac{\pi^2 r}{2}$

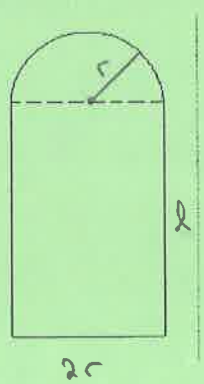
$r = \dfrac{3\pi}{\left(2\pi + \frac{\pi^2}{2}\right)} \approx .8401$

$\dfrac{dA}{dr}$: $-$ | $+$ ; $0$, $.8401$, $6$

$\boxed{\text{Min}: r = .8401 \quad \text{Max}: r = 6}$

2. A window consisting of a rectangle topped by a semicircle is to have an outer perimeter $P$. Find the radius of the semicircle if the area of the window is to be a maximum.

constant




$P = \pi r + 2l + 2r$

$l = \dfrac{P - \pi r - 2r}{2}$

$A = \frac{1}{2}\pi r^2 + 2rl$

$A = \frac{1}{2}\pi r^2 + 2r\left(\dfrac{P - \pi r - 2r}{2}\right)$

$A = \frac{1}{2}\pi r^2 + r(P - \pi r - 2r)$

$A = \frac{1}{2}\pi r^2 + Pr - \pi r^2 - 2r^2$

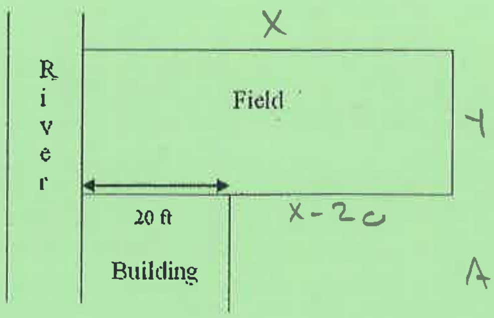
$A(r) = \left(\frac{1}{2}\pi - \pi - 2\right)r^2 + Pr$

$A(r) = \left(-\frac{1}{2}\pi - 2\right)r^2 + Pr$

$\dfrac{dA}{dr} = 2\left(-\frac{1}{2}\pi - 2\right)r + P$

$r = \dfrac{-P}{-\pi - 4} = \boxed{\dfrac{P}{\pi + 4}}$

3. A rectangular field as shown is to be bounded by a fence. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fencing. You can assume that fencing is not needed along the river and building.




$A = xy$

$1000 = y + 2x - 20$

$y = 1020 - 2x$

$A(x) = x(1020 - 2x)$

$A(x) = 1020x - 2x^2$

$\frac{dA}{dx} = 1020 - 4x = 0$

$$\boxed{\begin{aligned} x &= 255 \text{ ft} \\ y &= 510 \text{ ft} \end{aligned}}$$

4. A company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs $10 per square foot and the rest is made with material that costs $8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.


$C = 10 \cdot 2\pi r^2 + 8 \cdot 2\pi r h$

$5 = \pi r^2 h$

$h = \frac{5}{\pi r^2}$

$C(r) = 20\pi r^2 + \frac{80\pi r}{\pi r^2}$

$C(r) = 20\pi r^2 + \frac{80}{r}$

$\frac{dC}{dr} = 40\pi r - \frac{80}{r^2}$

$0 = 40\pi r - \frac{80}{r^2}$

$r^3 = \frac{80}{40\pi}$

$$\boxed{r = \sqrt[3]{\frac{2}{\pi}}}$$