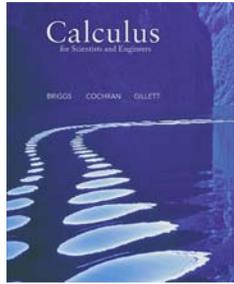


Chapter 10

Sequences and Infinite Series



10.1

An Overview

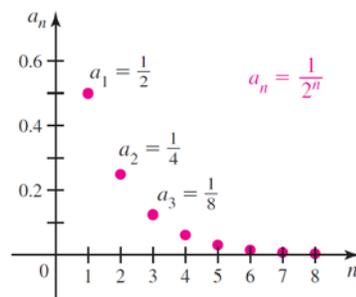
DEFINITION Sequence

A sequence $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \dots, a_n, \dots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \dots$, where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \dots$.

Figure 10.1



1

Figure 10.2

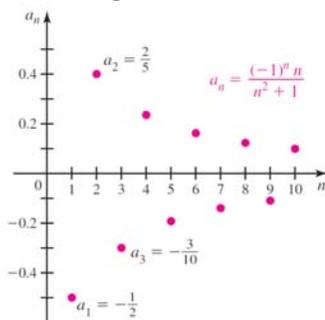
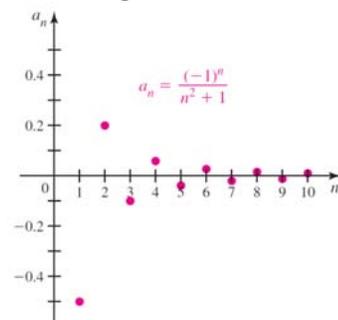


Figure 10.3



DEFINITION Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim_{n \rightarrow \infty} a_n = L$ exists, and the sequence **converges** to L . If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Figure 10.4

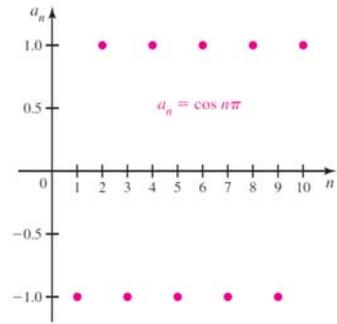


Figure 10.5

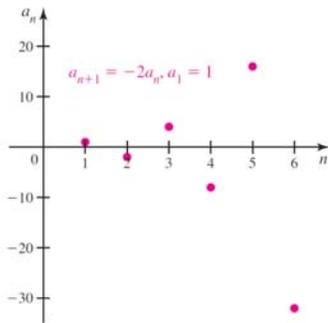


Table 10.1

n	a_n	n	a_n
1	2.000	8	3.992
2	3.556	9	3.995
3	3.857	10	3.996
4	3.938	11	3.997
5	3.968	12	3.998
6	3.982	13	3.998
7	3.988	14	3.999

Figure 10.6

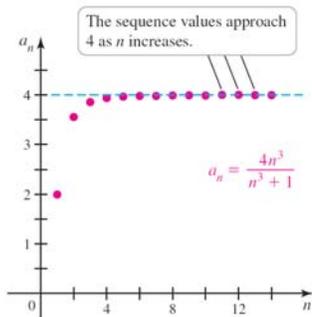
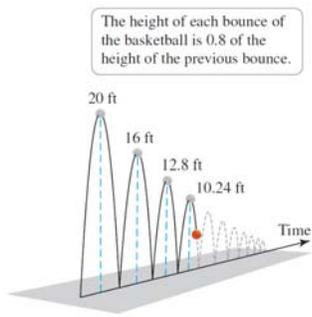
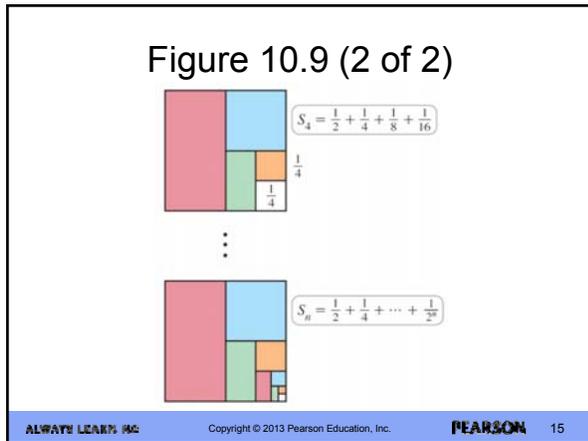
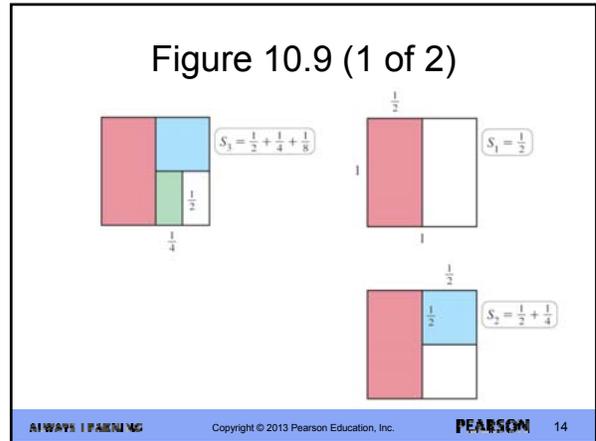
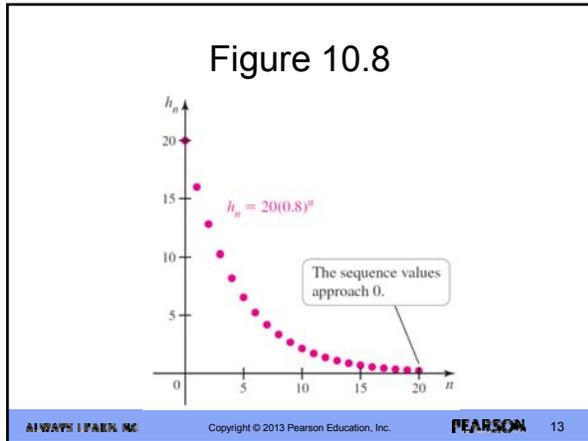


Figure 10.7





3

DEFINITION Infinite Series
 Given a set of numbers $\{a_1, a_2, a_3, \dots\}$, the sum

$$a_1 + a_2 + a_3 + \dots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**. Its sequence of **partial sums** $\{S_n\}$ has the terms

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k, \text{ for } n = 1, 2, 3, \dots$$

If the sequence of partial sums $\{S_n\}$ has a limit L , the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

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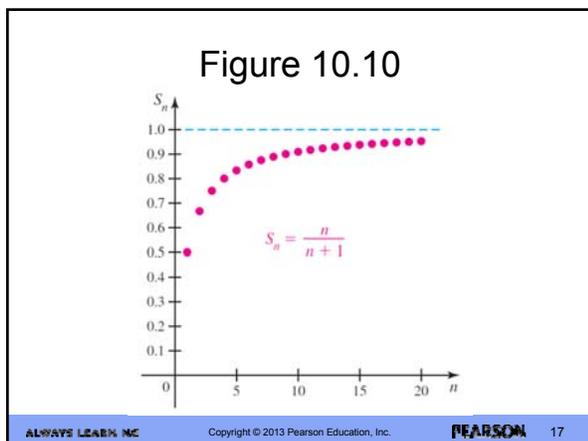


Table 10.2

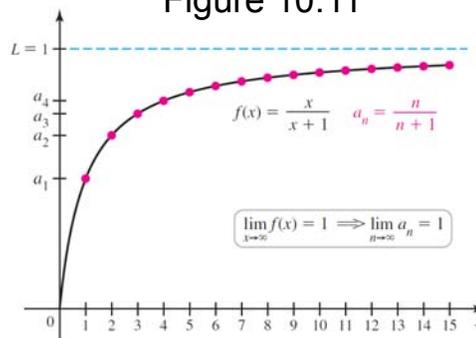
	Sequences/Series	Functions
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 0, 1, 2, 3, \dots$	Real numbers e.g., $\{x: x \geq 0\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=0}^n a_k$	$\int_0^n f(x) dx$
Accumulation over an infinite interval	$\sum_{k=0}^{\infty} a_k$	$\int_0^{\infty} f(x) dx$

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10.2

Sequences

Figure 10.11



THEOREM 10.1 Limits of Sequences from Limits of Functions
 Suppose f is a function such that $f(n) = a_n$ for all positive integers n .
 If $\lim_{x \rightarrow \infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L .

4

THEOREM 10.2 Limit Laws for Sequences

Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B , respectively. Then,

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n \rightarrow \infty} ca_n = cA$, where c is a real number
3. $\lim_{n \rightarrow \infty} a_n b_n = AB$
4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

Figure 10.12

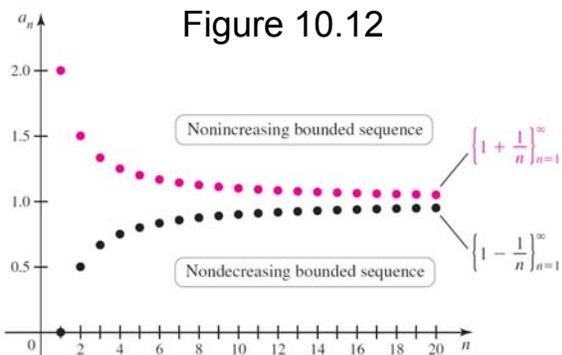


Figure 10.13

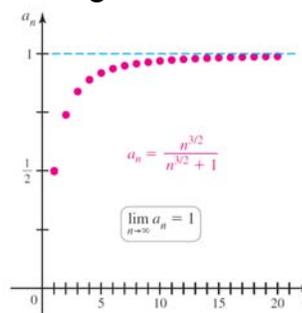


Figure 10.14

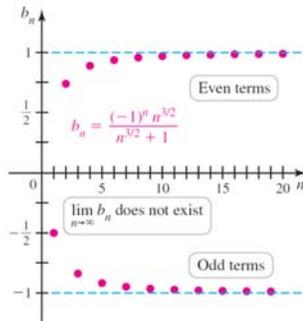


Figure 10.15

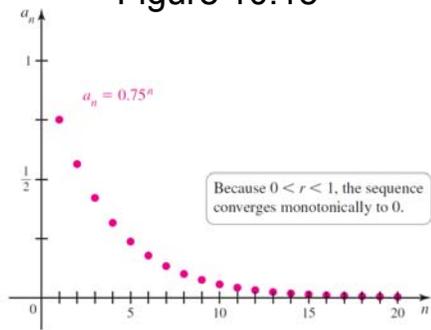


Figure 10.16

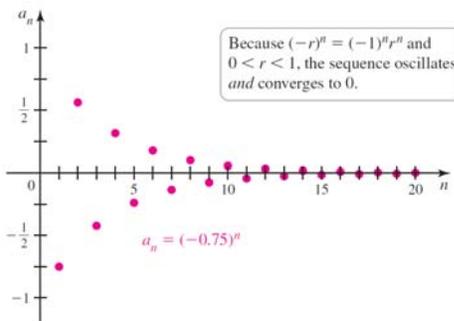
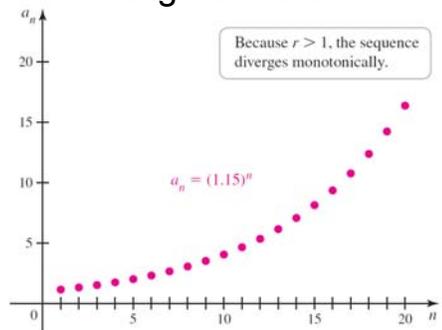
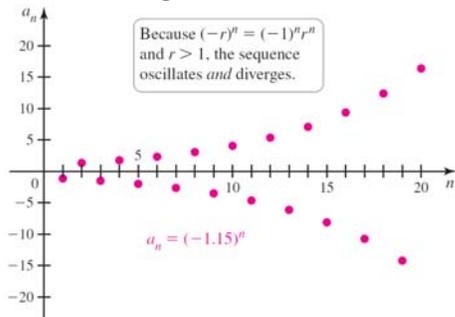


Figure 10.17



5

Figure 10.18



THEOREM 10.3 Geometric Sequences

Let r be a real number. Then,

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ converges or diverges monotonically. If $r < 0$, then $\{r^n\}$ converges or diverges by oscillation.

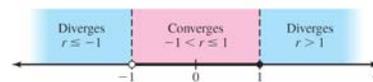
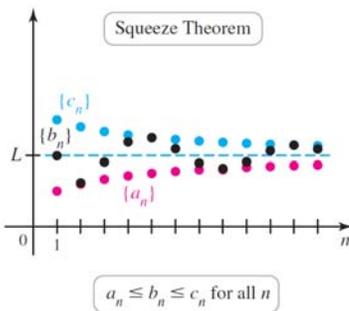


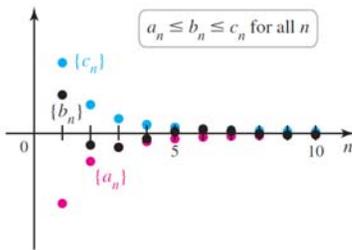
Figure 10.19



THEOREM 10.4 Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N . If $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ (Figure 10.19).

Figure 10.20



6

THEOREM 10.5 Bounded Monotonic Sequences
A bounded monotonic sequence converges.

Figure 10.21

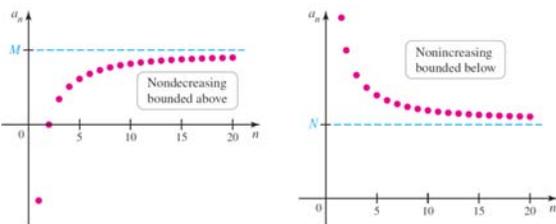
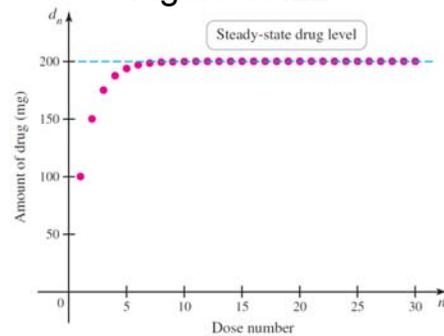


Figure 10.22



THEOREM 10.6 Growth Rates of Sequences

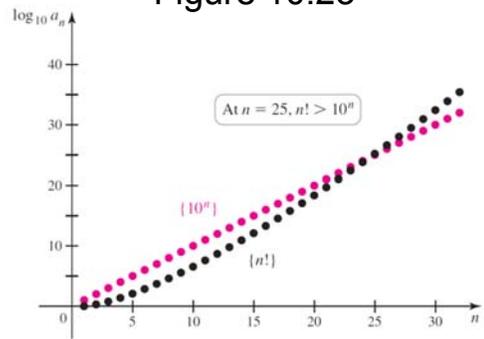
The following sequences are ordered according to increasing growth rates as $n \rightarrow \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$

and $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^n n\} \ll \{n^p\} \ll \{n^p \ln^n n\} \ll \{n^{p+1}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s , and $b > 1$.

Figure 10.23



DEFINITION Limit of a Sequence

The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any tolerance $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

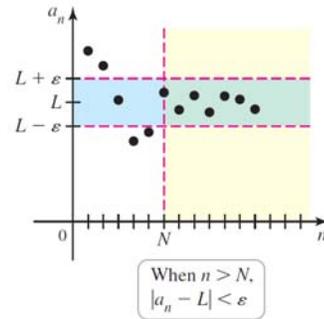
$$|a_n - L| < \varepsilon \quad \text{whenever } n > N.$$

If the **limit of a sequence** is L , we say the sequence **converges** to L , written

$$\lim_{n \rightarrow \infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.

Figure 10.24



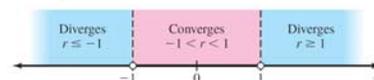
7

10.3

Infinite Series

THEOREM 10.7 Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.



10.4

The Divergence and Integral Tests

THEOREM 10.8 Divergence Test

If $\sum a_k$ converges, then $\lim_{k \rightarrow \infty} a_k = 0$. Equivalently, if $\lim_{k \rightarrow \infty} a_k \neq 0$, then the series diverges.

Figure 10.25

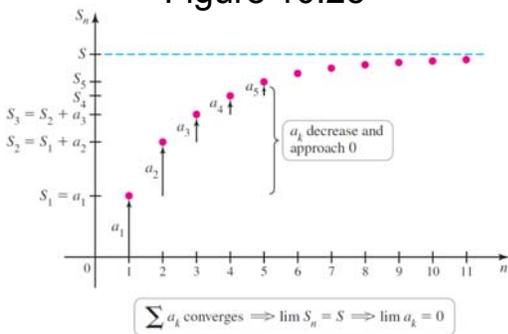


Table 10.3

n	S_n	n	S_n
10^3	≈ 7.49	10^{10}	≈ 23.60
10^4	≈ 9.79	10^{20}	≈ 46.63
10^5	≈ 12.09	10^{30}	≈ 69.65
10^6	≈ 14.39	10^{40}	≈ 92.68

8

Figure 10.26

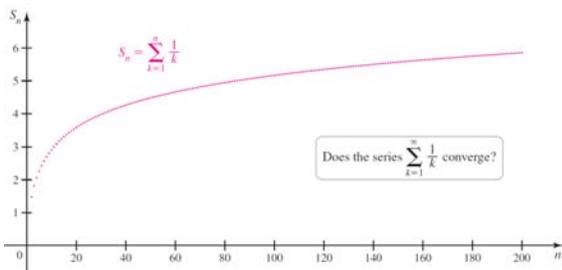
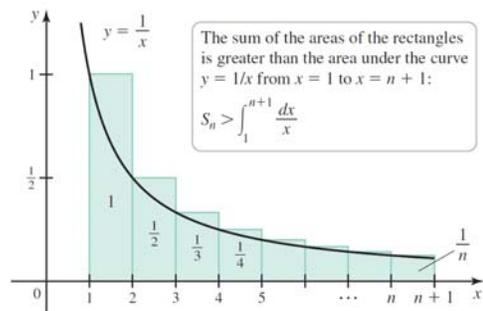


Figure 10.27



THEOREM 10.9 Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ diverges—even though the terms of the series approach zero.

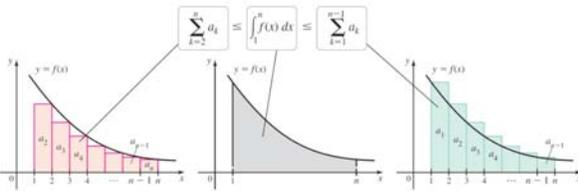
THEOREM 10.10 Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \dots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x) \, dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not*, in general, equal to the value of the series.

Figure 10.28



9

THEOREM 10.11 Convergence of the p -Series

The p -series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.

Figure 10.29

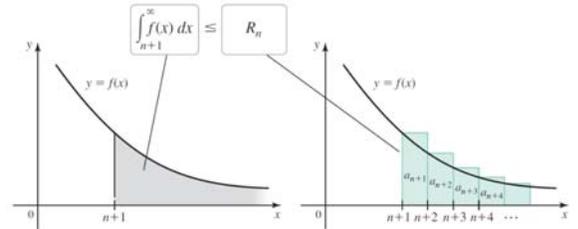
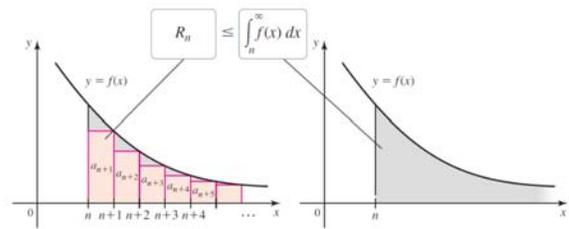


Figure 10.30



THEOREM 10.12 Estimating Series with Positive Terms

Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \dots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^n a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n \leq \int_n^{\infty} f(x) dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x) dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x) dx.$$

Figure 10.31 (a)

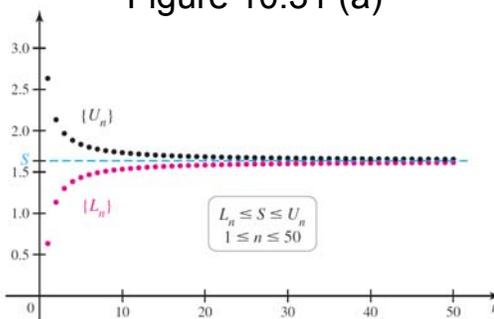
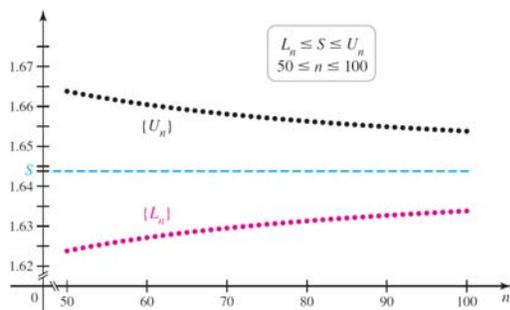


Figure 10.31 (b)

**THEOREM 10.13 Properties of Convergent Series**

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c \sum a_k = cA$.
2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B . The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
3. Whether a series converges does not depend on a finite number of terms added to or removed from the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the value of a convergent series does change if nonzero terms are added or deleted.

10.5

The Ratio, Root, and Comparison Tests

THEOREM 10.14 The Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k}$.

1. If $0 \leq r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

THEOREM 10.15 The Root Test

Let $\sum a_k$ be an infinite series with nonnegative terms and let $\rho = \lim_{k \rightarrow \infty} \sqrt[k]{a_k}$.

1. If $0 \leq \rho < 1$, the series converges.
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

THEOREM 10.16 Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Figure 10.32

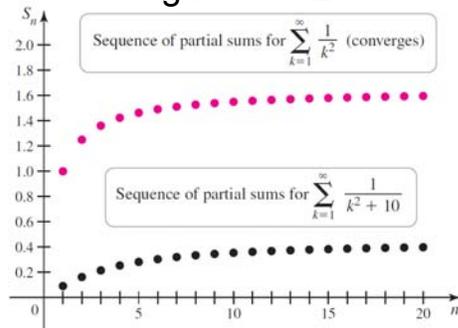
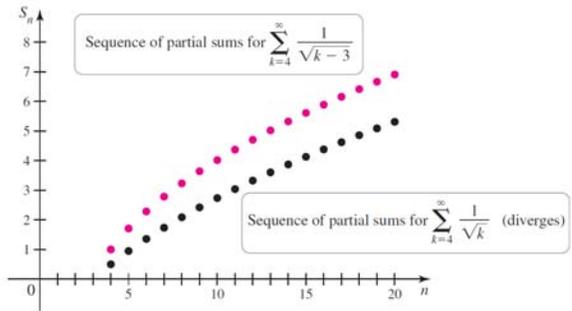


Figure 10.33



1

THEOREM 10.17 The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

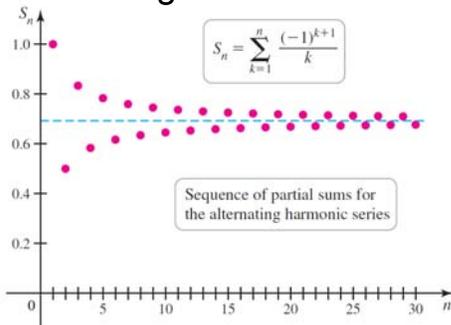
$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L.$$

1. If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

10.6

Alternating Series

Figure 10.34

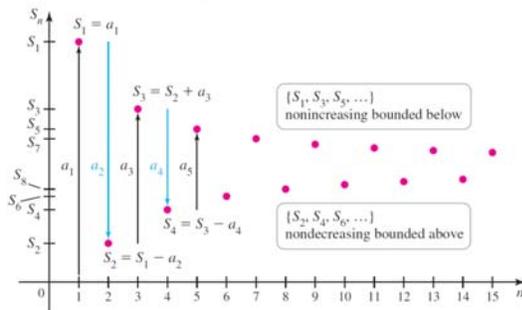


THEOREM 10.18 The Alternating Series Test

The alternating series $\sum (-1)^{k+1} a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim_{k \rightarrow \infty} a_k = 0$.

Figure 10.35



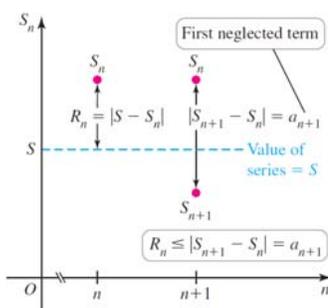
1

THEOREM 10.19 Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ diverges).

Figure 10.36



THEOREM 10.20 Remainder in Alternating Series

Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ by the sum of its first n terms. Then $R_n \leq a_{n+1}$.

In other words, the remainder is less than or equal to the magnitude of the first neglected term.

DEFINITION Absolute and Conditional Convergence

Assume the infinite series $\sum a_k$ converges. The series $\sum a_k$ converges absolutely if the series $\sum |a_k|$ converges. Otherwise, the series $\sum a_k$ converges conditionally.

THEOREM 10.21 Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Figure 10.37

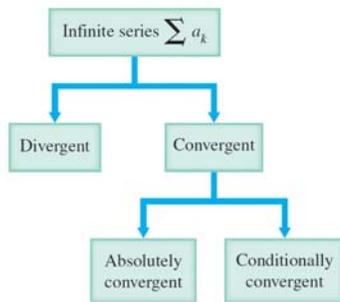


Table 10.4 (1 of 2)

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=1}^{\infty} ar^{k-1}, a \neq 0$	$ r < 1$	$ r \geq 1$	If $ r < 1$, then $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k \rightarrow \infty} a_k \neq 0$	Cannot be used to prove convergence.
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x) dx < \infty$	$\int_1^{\infty} f(x) dx$ does not exist	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$	$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$	$\lim_{k \rightarrow \infty} \sqrt[k]{a_k} < 1$	$\lim_{k \rightarrow \infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = 1$

Table 10.4 (2 of 2)

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0, b_k > 0$	$0 \leq \lim_{k \rightarrow \infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0, 0 < a_{k+1} \leq a_k$	$\lim_{k \rightarrow \infty} a_k = 0$	$\lim_{k \rightarrow \infty} a_k \neq 0$	Remainder R_n satisfies $R_n \leq a_{n+1}$.
Absolute Convergence	$\sum_{k=1}^{\infty} a_k, a_k$ arbitrary	$\sum_{k=1}^{\infty} a_k $ converges		Applies to arbitrary series.