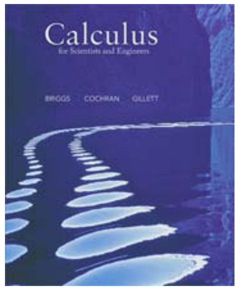
Chapter 2

Limits



2.1

The Idea of Limits

1

Figure 2.1

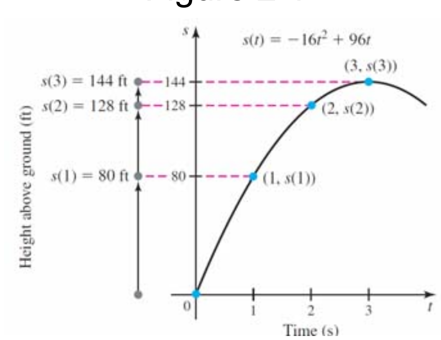


Figure 2.2 (a)

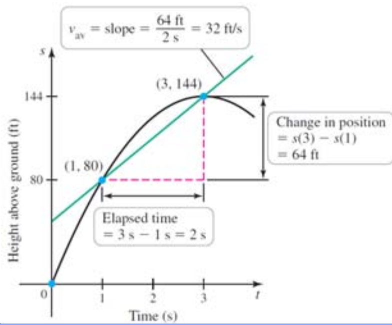
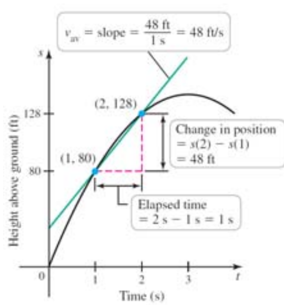


Figure 2.2 (b)



2

Figure 2.3

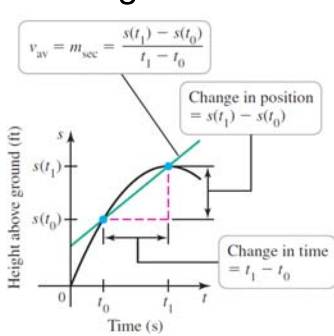


Table 2.1

Time interval	Average velocity
[1, 2]	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

Figure 2.4

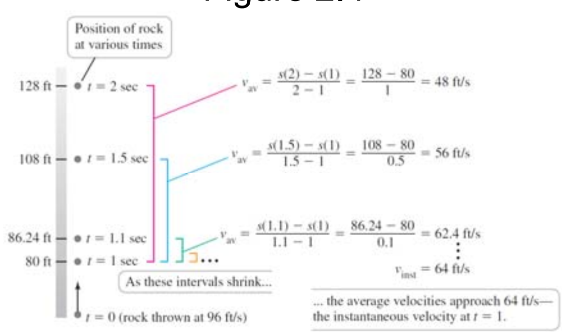


Figure 2.5

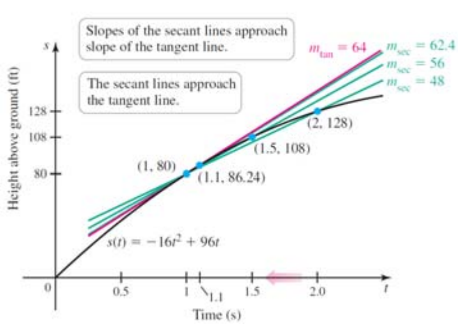


Figure 2.6 (1 of 3)

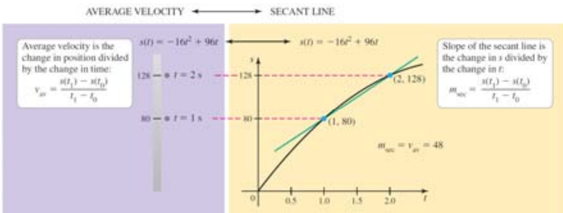


Figure 2.6 (2 of 3)

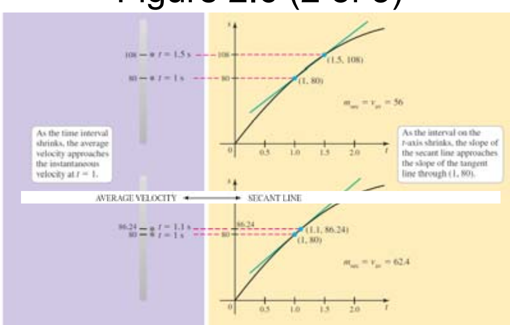
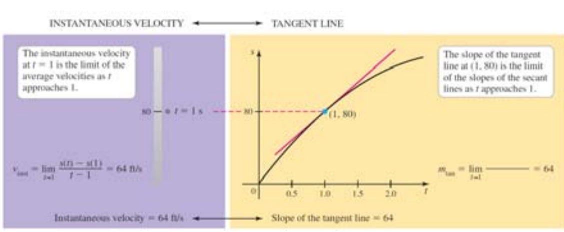


Figure 2.6 (3 of 3)



2.2

Definitions of Limits

DEFINITION Limit of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

Figure 2.7

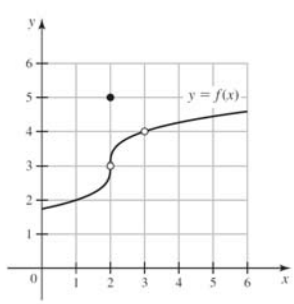
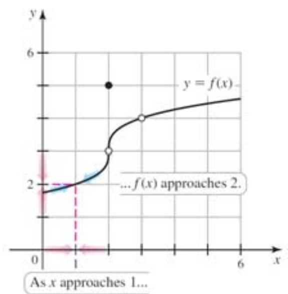
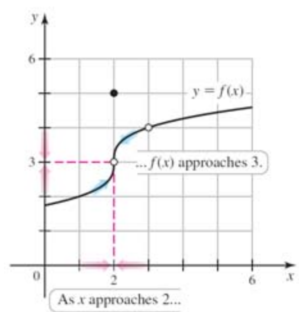


Figure 2.8



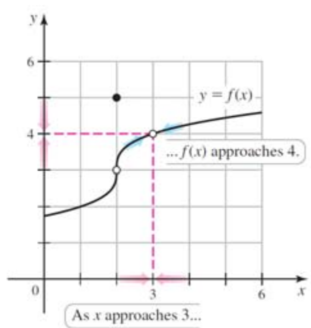
ALWAYS LEARNING Copyright © 2013 Pearson Education, Inc. PEARSON 16

Figure 2.9



ALWAYS LEARNING Copyright © 2013 Pearson Education, Inc. PEARSON 17

Figure 2.10



ALWAYS LEARNING Copyright © 2013 Pearson Education, Inc. PEARSON 18

Table 2.2

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = \frac{\sqrt{x} - 1}{x - 1}$	0.5131670	0.5012563	0.5001251	0.5000125	0.4999875	0.4998751	0.4987562	0.4880885

DEFINITION One-Sided Limits

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Figure 2.11 (a & b)

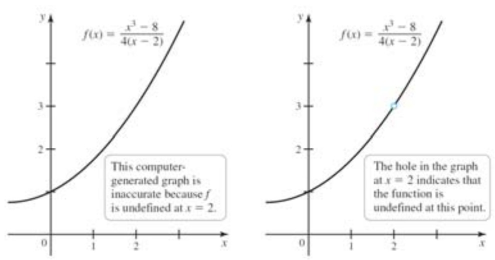


Figure 2.12 (a & b)

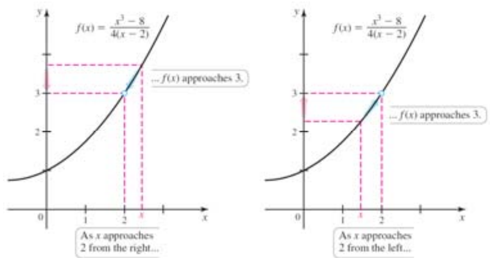
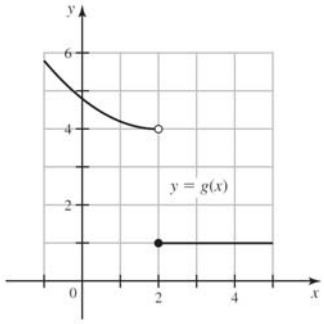


Table 2.3

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
	→	→	→	→ 2 ←	←	←	←	
$f(x) = \frac{x^3 - 8}{4(x - 2)}$	2.8525	2.985025	2.99850025	2.99985000	3.00015000	3.00150025	3.015025	3.1525

THEOREM 2.1 Relationship Between One-Sided and Two-sided Limits
Assume f is defined for all x near a except possibly at a. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Figure 2.13



 25

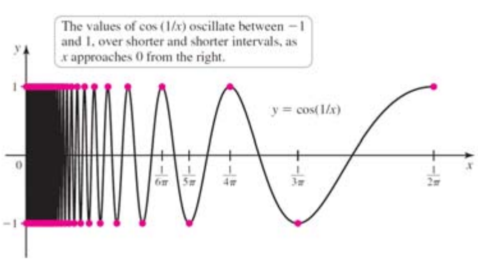
Table 2.4

x	$\cos(1/x)$
0.001	0.56238
0.0001	−0.95216
0.00001	−0.99936
0.000001	0.93675
0.0000001	−0.90727
0.00000001	−0.36338

We might *incorrectly* conclude that $\cos(1/x)$ approaches −1 as x approaches 0 from the right. (first three values)

 26

Figure 14

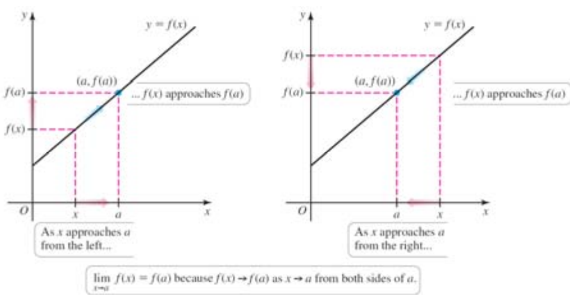


 27

2.3

Techniques for Computing Limits

Figure 2.15



THEOREM 2.2 Limits of Linear Functions
Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

THEOREM 2.3 Limit Laws

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

1. **Sum** $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Difference** $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. **Constant multiple** $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. **Product** $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$
5. **Quotient** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
6. **Power** $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. **Fractional power** $\lim_{x\to a}[f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms

THEOREM 2.4 Limits of Polynomial and Rational Functions

Assume p and q are polynomials and a is a constant.

a. Polynomial functions: $\lim_{x\to a} p(x) = p(a)$

b. Rational functions: $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$

THEOREM 2.3 (CONTINUED) Limit Laws for One-Sided Limits

Laws 1–6 hold with $\lim_{x\to a}$ replaced by $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$. Law 7 is modified as follows. Assume $m > 0$ and $n > 0$ are integers.

7. Fractional power

a. $\lim_{x\to a^+}[f(x)]^{n/m} = \left[\lim_{x\to a^+} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x > a$, if m is even and n/m is reduced to lowest terms

b. $\lim_{x\to a^-}[f(x)]^{n/m} = \left[\lim_{x\to a^-} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x < a$, if m is even and n/m is reduced to lowest terms

Figure 2.16

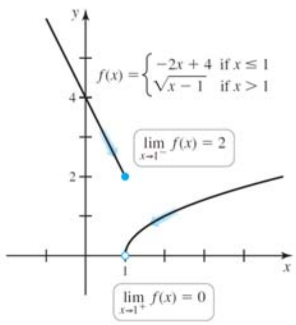


Figure 2.17

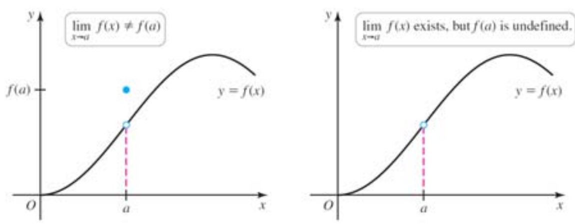


Figure 2.18

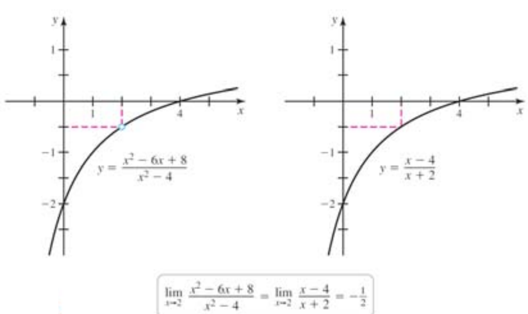
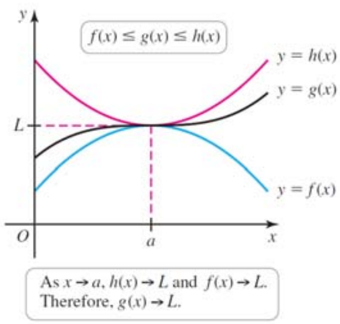


Figure 2.19



THEOREM 2.5 The Squeeze Theorem
Assume the functions f, g, and h satisfy $f(x) \le g(x) \le h(x)$ for all values of x near a, except possibly at a. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Figure 2.20

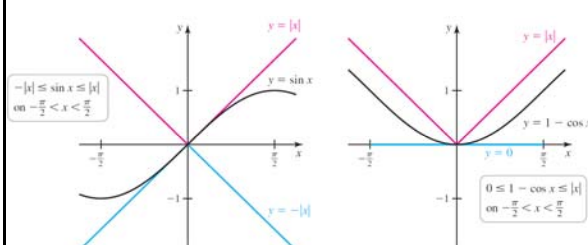
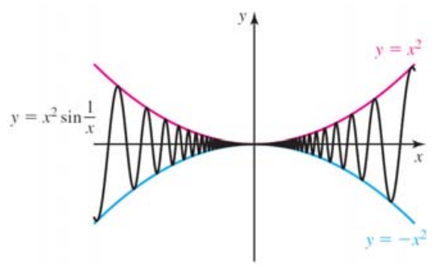


Figure 2.21



2.4

Infinite Limits

Table 2.5

x	$f(x) = 1/x^2$
± 0.1	100
± 0.01	10,000
± 0.001	1,000,000
$\downarrow$	$\downarrow$
0	∞

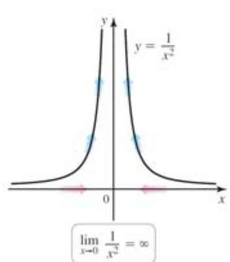


Table 2.6

x	$f(x) = 1/x^2$
10	0.01
100	0.0001
1000	0.000001
↓	↓
∞	0

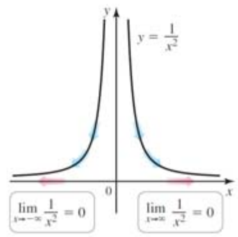


Figure 2.22

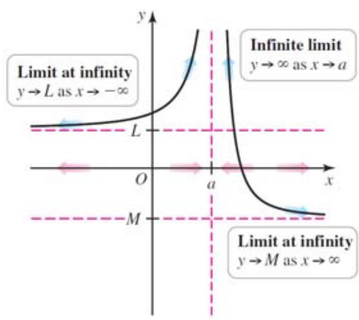


Figure 2.23 (a)

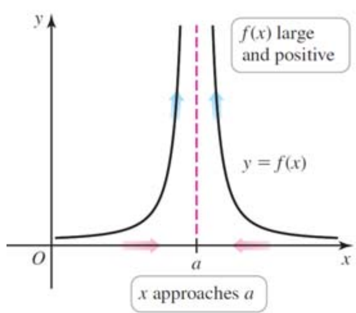
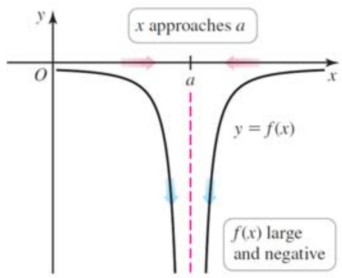


Figure 2.23 (b)



DEFINITION Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a (Figure 2.23a), we write

$$\lim_{x \to a} f(x) = \infty.$$

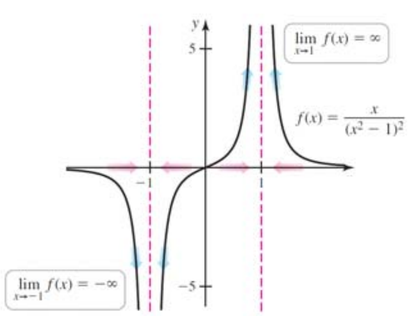
We say the limit of $f(x)$ as x approaches a is infinity.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a (Figure 2.23b), we write

$$\lim_{x \to a} f(x) = -\infty.$$

In this case, we say the limit of $f(x)$ as x approaches a is negative infinity. In both cases, the limit does not exist.

Figure 2.24



DEFINITION One-Sided Infinite Limits

Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write $\lim_{x \to a^+} f(x) = \infty$ (Figure 2.25a).
The one-sided infinite limits $\lim_{x \to a^+} f(x) = -\infty$ (Figure 2.25b), $\lim_{x \to a^-} f(x) = \infty$ (Figure 2.25c), and $\lim_{x \to a^-} f(x) = -\infty$ (Figure 2.25d) are defined analogously.

Figure 2.25 (a & b) continued…

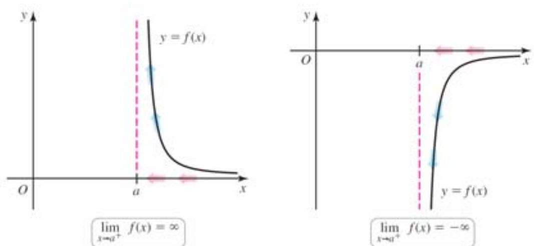
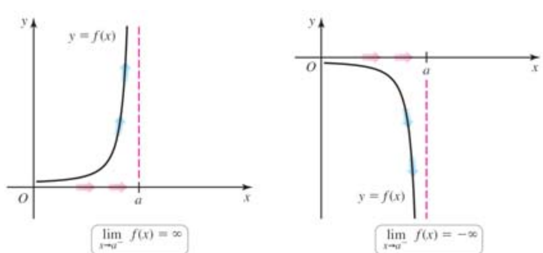


Figure 2.25 (c & d)



DEFINITION Vertical Asymptote

If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a^+} f(x) = \pm\infty$, or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

Figure 2.26

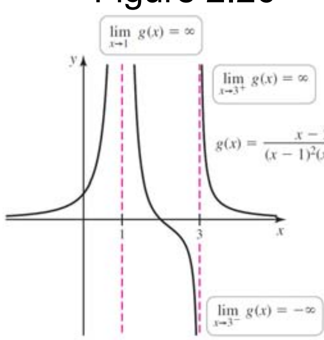


Table 2.7

x	$\frac{5+x}{x}$
0.01	$\frac{5.01}{0.01} = 501$
0.001	$\frac{5.001}{0.001} = 5001$
0.0001	$\frac{5.0001}{0.0001} = 50{,}001$
$\downarrow$	$\downarrow$
0^+	∞

Figure 2.27

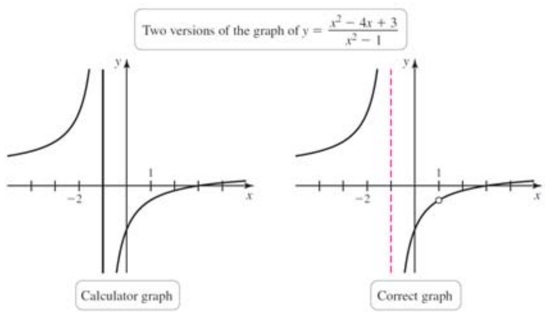
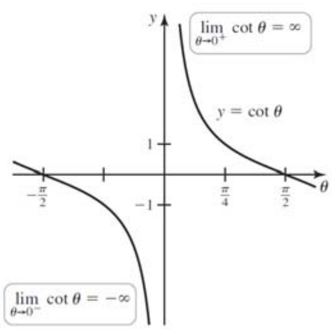


Figure 2.28



2.5

Limits at Infinity

Figure 2.29

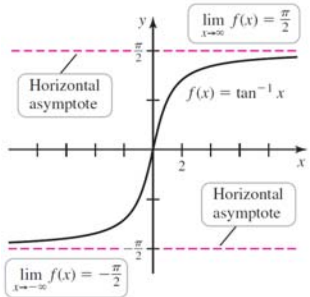
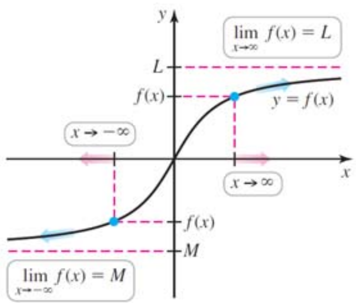


Figure 2.30



DEFINITION Limits at Infinity and Horizontal Asymptotes

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case the line $y = L$ is a **horizontal asymptote** of f (Figure 2.30). The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously. When the limit exists, the horizontal asymptote is $y = M$.

Figure 2.31

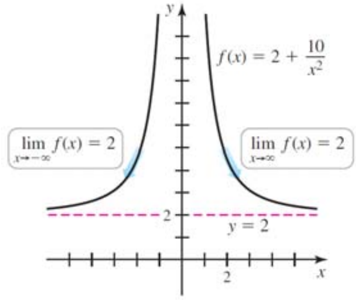


Figure 2.32

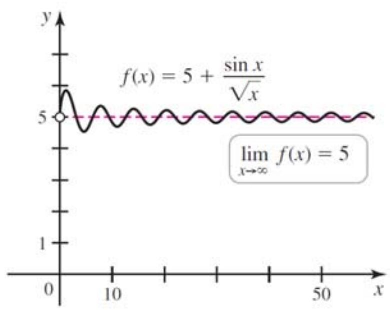
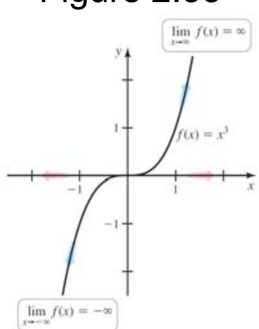


Figure 2.33



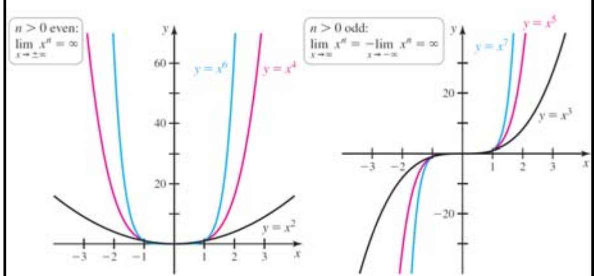
DEFINITION Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x \to \infty} f(x) = \infty.$$

The limits $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$ are defined similarly.

Figure 2.34



THEOREM 2.6 Limits at Infinity of Powers and Polynomials

Let n be a positive integer and let p be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$.

1. $\lim_{x \to \pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$.
4. $\lim_{x \to \pm\infty} p(x) = \infty$ or $-\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Figure 2.35

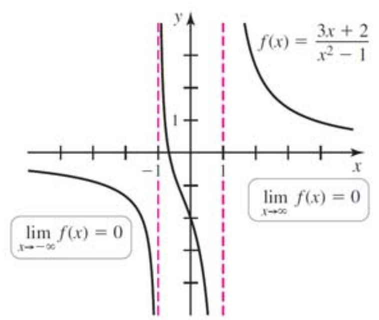


Figure 2.36

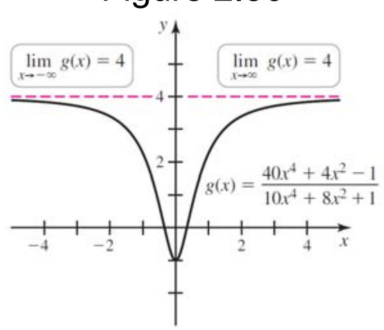
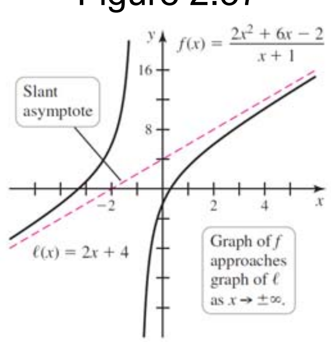


Figure 2.37



THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0 \quad \text{and}$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0,$$

with $a_m \neq 0$ and $b_n \neq 0$.

a. If $m < n$, then $\lim_{x \to \pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

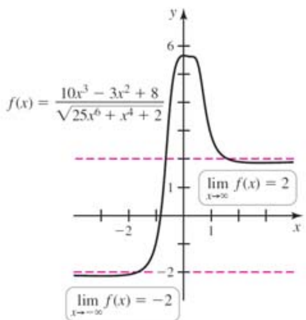
b. If $m = n$, then $\lim_{x \to \pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote of f.

c. If $m > n$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

d. If $m = n + 1$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, f has no horizontal asymptote, but f has a slant asymptote.

e. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Figure 2.38



2.6

Continuity

Figure 2.39 (a)

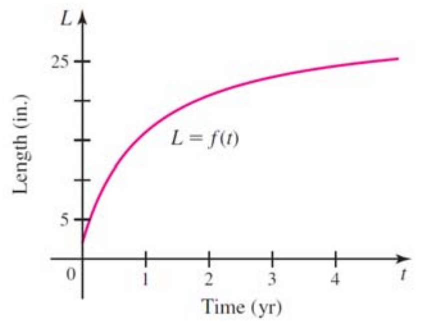


Figure 2.39 (b)

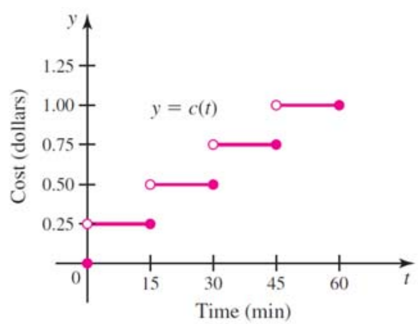
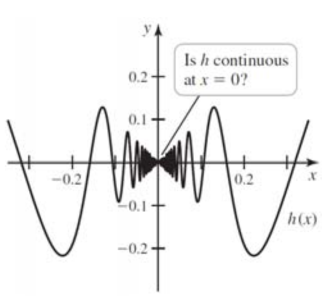


Figure 2.40



$$h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

DEFINITION Continuity at a Point

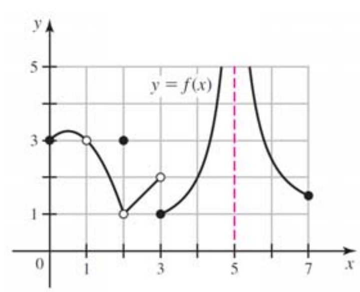
A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, then a is a point of discontinuity.

Continuity Checklist

In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Figure 2.41



THEOREM 2.8 Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$

b. $f - g$

c. cf

d. fg

e. f/g, provided $g(a) \neq 0$

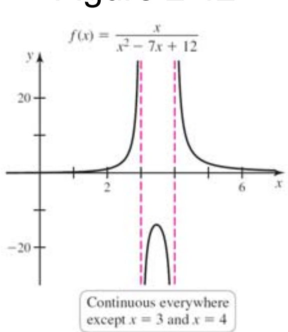
f. $(f(x))^n$

THEOREM 2.9 Polynomial and Rational Functions

a. A polynomial function is continuous for all x.

b. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Figure 2.42



THEOREM 2.10 Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Figure 2.43 (a)

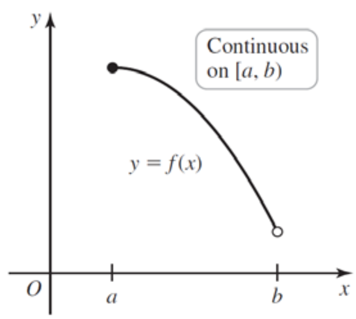
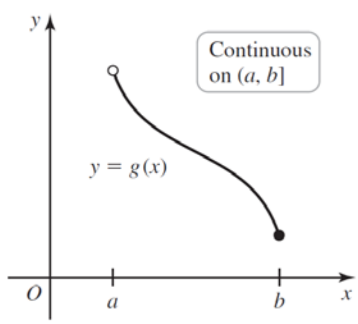


Figure 2.43 (b)



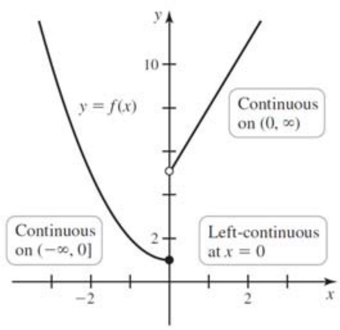
DEFINITION **Continuity at Endpoints**

A function f is **continuous from the left** (or **left-continuous**) at a if $\lim_{x \to a^-} f(x) = f(a)$ and f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$.

DEFINITION **Continuity on an Interval**

A function f is **continuous on an interval** I if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

Figure 2.44



THEOREM 2.11 Continuity of Functions with Roots
Assume that m and n are positive integers with no common factors. If m is an odd integer, then $[f(x)]^{n/m}$ is continuous at all points at which f is continuous.
If m is even, then $[f(x)]^{n/m}$ is continuous at all points a at which f is continuous and $f(a) > 0$.

Figure 2.45

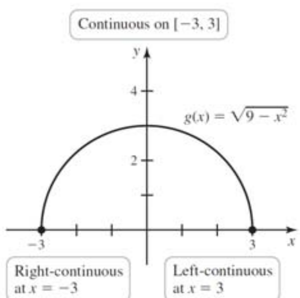


Figure 2.46

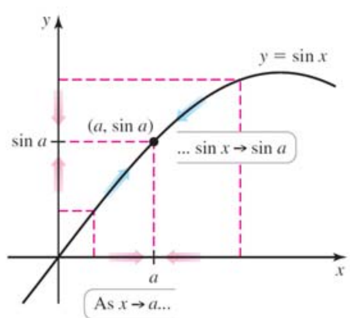
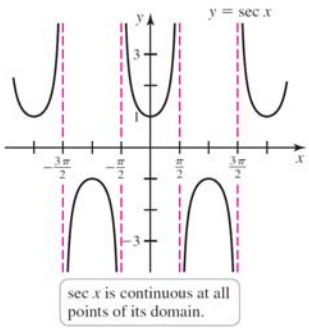


Figure 2.47



THEOREM 2.12 Continuity of Trigonometric Functions
The functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous at all points of their domains.

Figure 2.48

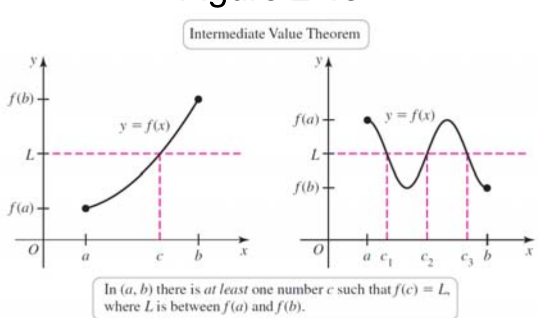
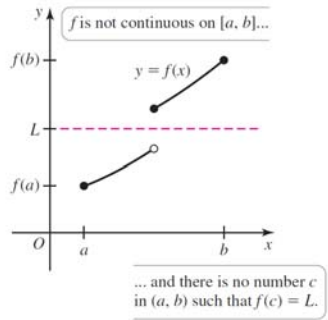


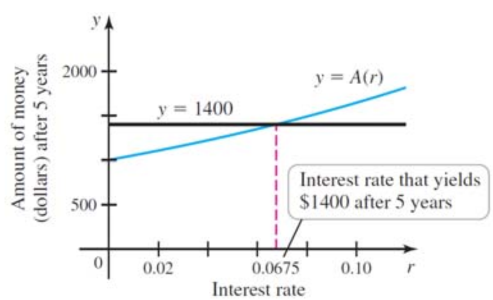
Figure 2.49



THEOREM 2.13 The Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number between $f(a)$ and $f(b)$. Then there is at least one number c in (a, b) satisfying $f(c) = L$.

32

Figure 2.50



2.7

Precise Definitions of Limits

Figure 2.51

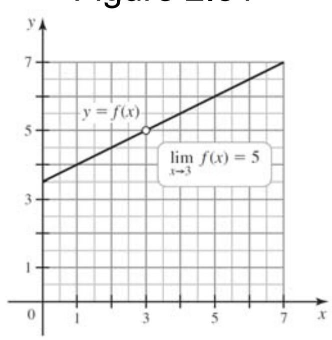


Figure 2.52 (a)

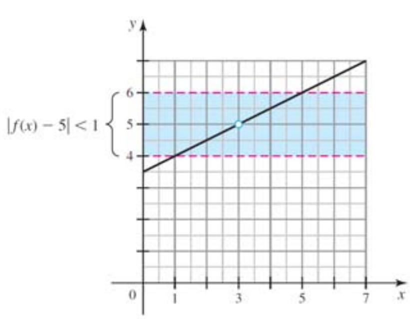


Figure 2.52 (b)

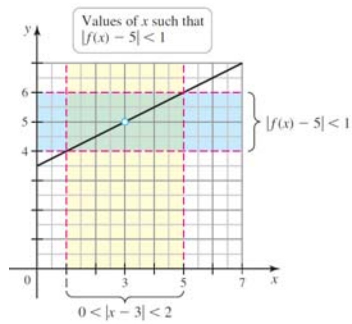


Figure 2.53 continued…

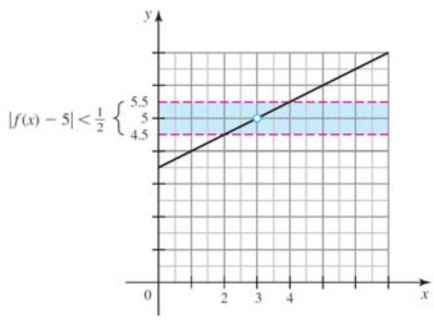


Figure 2.53

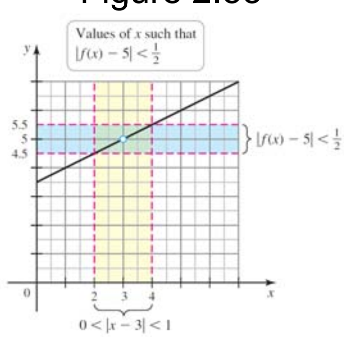


Figure 2.54

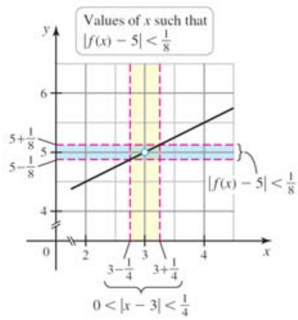
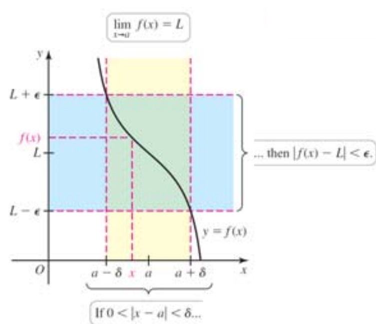


Figure 2.55



DEFINITION Limit of a Function

Assume that $f(x)$ exists for all x in some open interval containing a, except possibly at a. We say that the **limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x \to a} f(x) = L,$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Figure 2.56

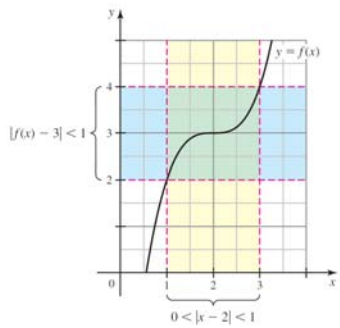


Figure 2.57

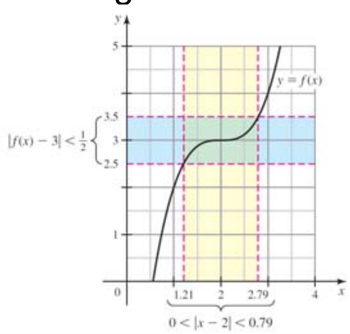


Figure 2.58

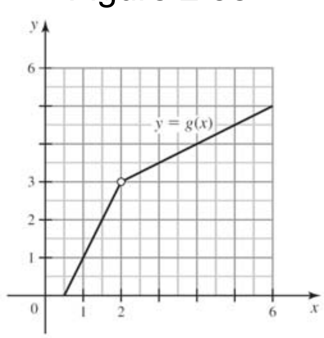


Figure 2.59 (a)

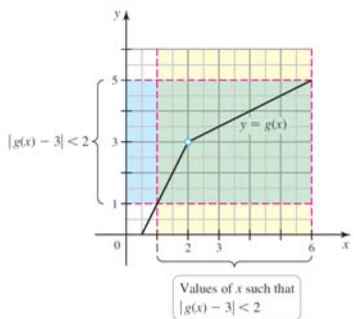


Figure 2.59 (b)

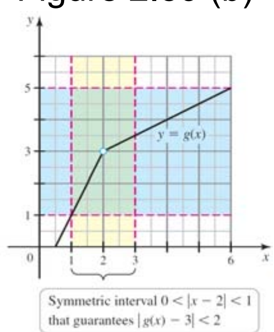


Figure 2.60 (a)

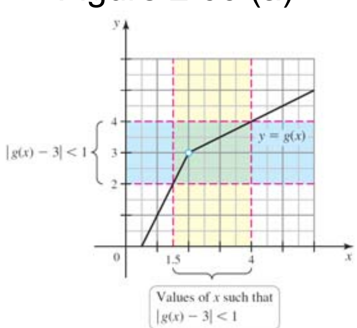
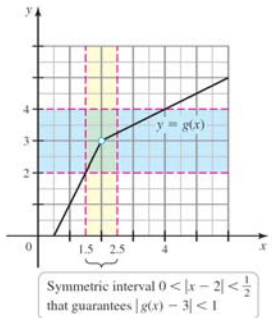


Figure 2.60 (b)



Steps for proving that $\lim_{x \to a} f(x) = L$

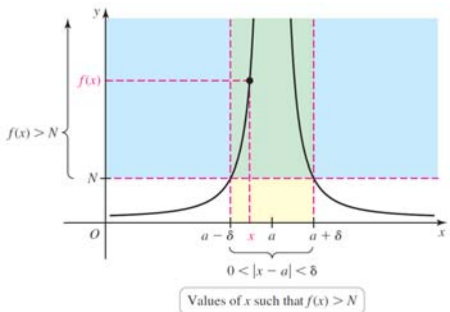
1. **Find δ.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

DEFINITION Two-Sided Infinite Limit

The **infinite limit** $\lim_{x \to a} f(x) = \infty$ means that for any positive number N there exists a corresponding $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Figure 2.61



Steps for proving that $\lim_{x \to a} f(x) = \infty$

1. **Find δ.** Let N be an arbitrary positive number. Use the statement $f(x) > N$ to find an inequality of the form $|x - a| < \delta$, where δ depends only on N.
2. **Write a proof.** For any $N > 0$, assume $0 < |x - a| < \delta$ and use the relationship between N and δ found in Step 1 to prove that $f(x) > N$.