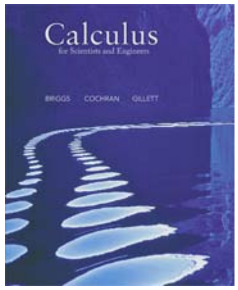
Chapter 3



Derivatives

3.1

Introducing the Derivative

1

Figure 3.1

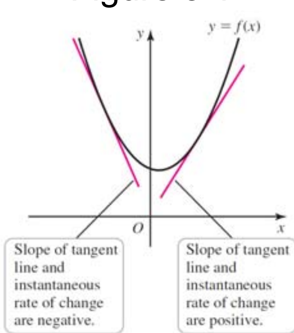


Figure 3.2

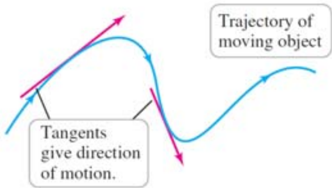


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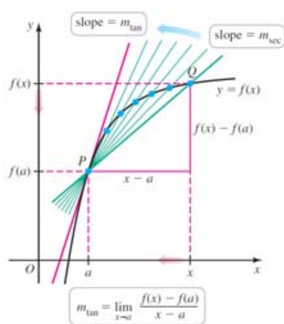
Figure 3.3



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DEFINITION Rates of Change and the Tangent Line

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at $x = a$ is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. This **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{tan}(x - a).$$

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Figure 3.4

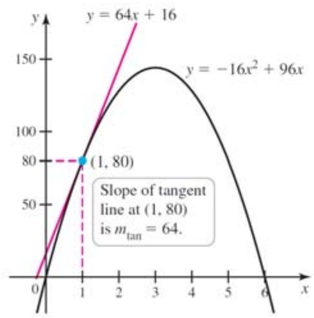
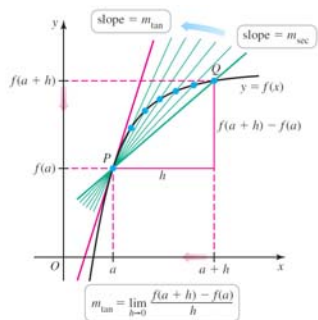


Figure 3.5



3

ALTERNATIVE DEFINITION Rates of Change and the Tangent Line

The **average rate of change** in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:

$$m_{sec} = \frac{f(a + h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at $x = a$ is

$$m_{tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \quad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Figure 3.6

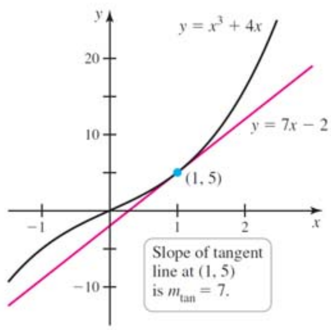


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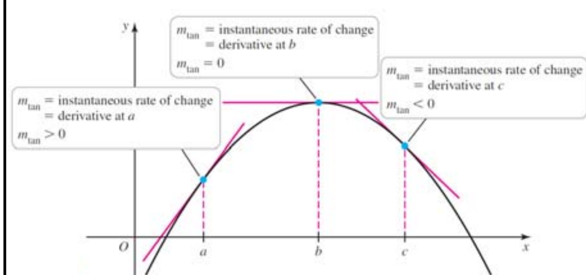
Figure 3.7



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DEFINITION The Derivative

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

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Figure 3.8

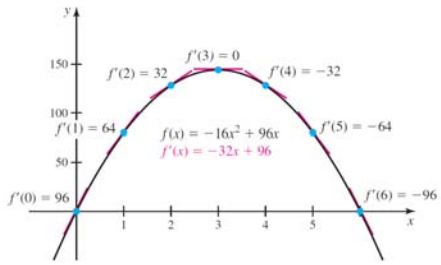

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Figure 3.9

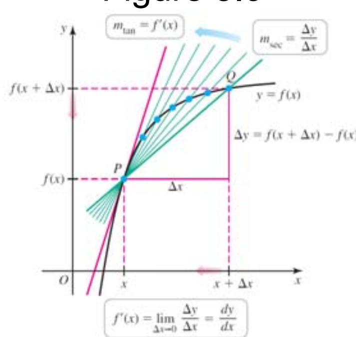

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Figure 3.10

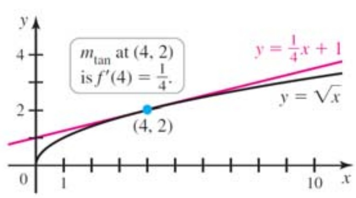

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Figure 3.11

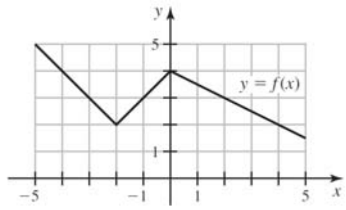

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Figure 3.12

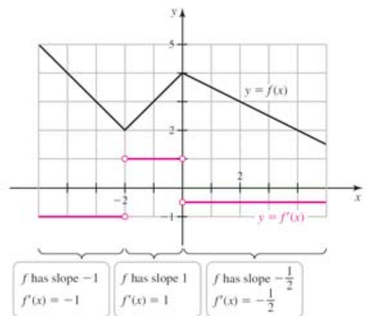

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Figure 3.13

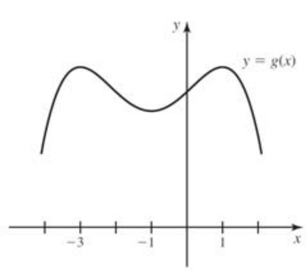

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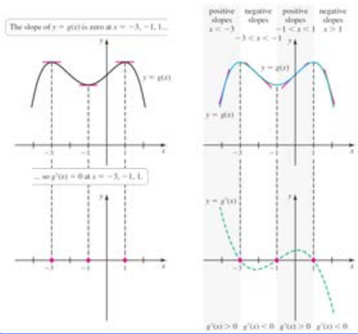
Figure 3.14



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7

THEOREM 3.1 Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.

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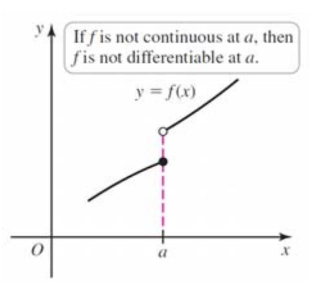
Figure 3.15



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THEOREM 3.1 (ALTERNATIVE VERSION) Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.

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Figure 3.16

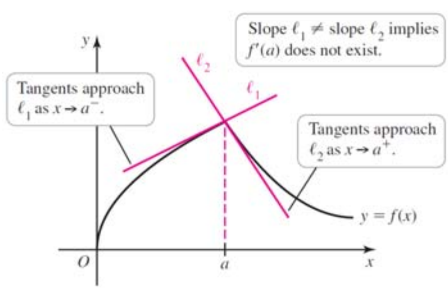

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Figure 3.17 (a)

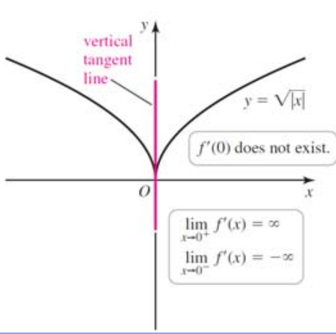

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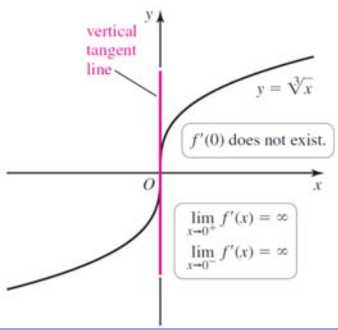
Figure 3.17 (b)



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When Is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a (Figure 3.15).

b. f has a corner at a (Figure 3.16).

c. f has a vertical tangent at a (Figure 3.17).

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Figure 3.18

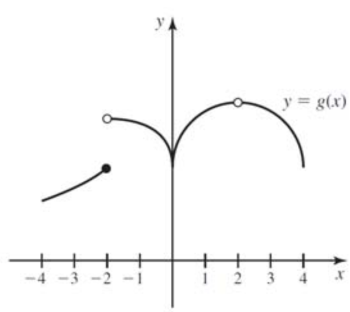

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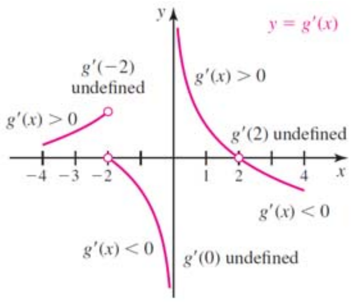
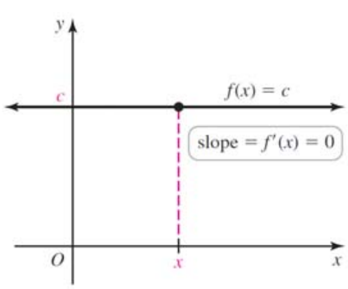
Figure 3.19



3.2

Rules of Differentiation

Figure 3.20



THEOREM 3.2 Constant Rule

If c is a real number, then $\frac{d}{dx}(c) = 0$.

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THEOREM 3.3 Power Rule

If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

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THEOREM 3.4 Constant Multiple Rule

If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x).$$

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THEOREM 3.5 Sum Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

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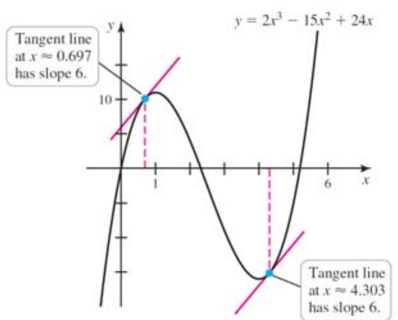
Figure 3.21



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DEFINITION Higher-Order Derivatives
Assuming f can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = f^{(2)}(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 1$, the ***n*th derivative** is

$$f^{(n)}(x) = \frac{d^nf}{dx^n} = \frac{d}{dx}(f^{(n-1)}(x)).$$

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3.3

The Product and Quotient Rules

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13

THEOREM 3.6 Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

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THEOREM 3.7 The Quotient Rule
If f and g are differentiable at x, then the derivative of f/g at x exists, provided $g(x) \neq 0$, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

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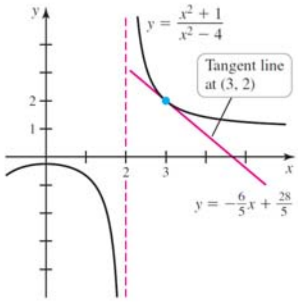
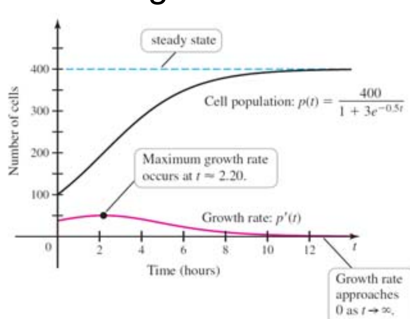
Figure 3.22



THEOREM 3.8 Extended Power Rule
If n is any integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Figure 3.23



3.4

Derivatives of Trigonometric Functions

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THEOREM 3.9 Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

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Table 3.1

x	$\frac{\sin x}{x}$
± 0.1	0.9983341665
± 0.01	0.9999833334
± 0.001	0.9999998333

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Figure 3.24

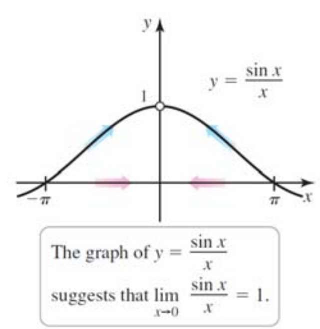

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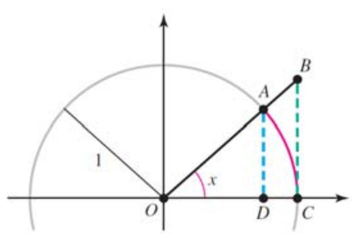
Figure 3.25



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THEOREM 3.10 Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

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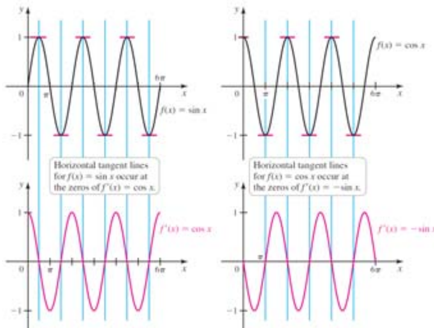
Figure 3.26 (a & b)



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THEOREM 3.11 **Derivatives of the Trigonometric Functions**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

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3.5

Derivatives as Rates of Change

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Figure 3.27

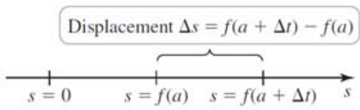

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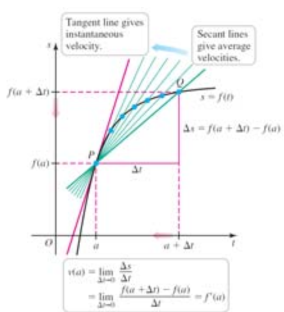
Figure 3.28



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DEFINITION Average and Instantaneous Velocity

Let $s = f(t)$ be the position function of an object moving along a line. The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:

$$v_{av} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The **instantaneous velocity** at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

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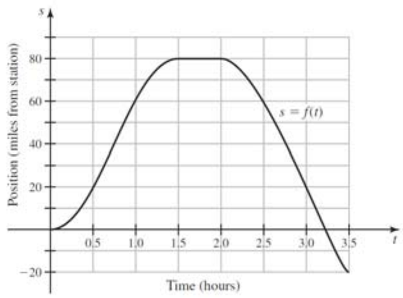
Figure 3.29



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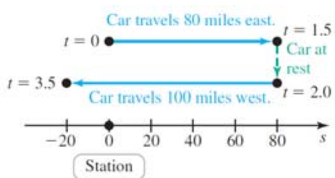

DEFINITION Velocity, Speed, and Acceleration

Suppose an object moves along a line with position $s = f(t)$. Then

the **velocity** at time t is $v = \frac{ds}{dt} = f'(t)$,

the **speed** at time t is $|v| = |f'(t)|$,

the **acceleration** at time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$.

Figure 3.31

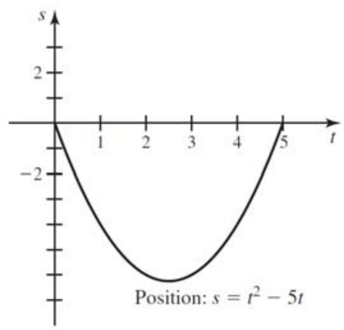

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Figure 3.32

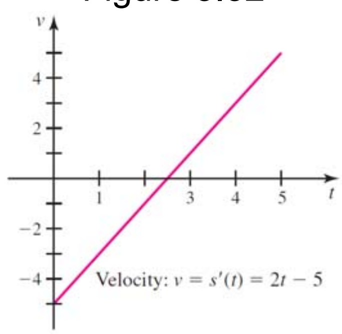

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Figure 3.33

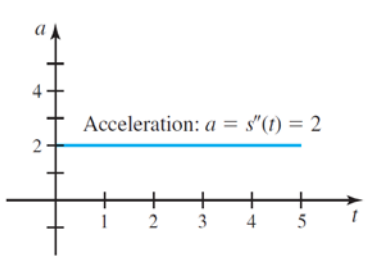

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Figure 3.34 (a)

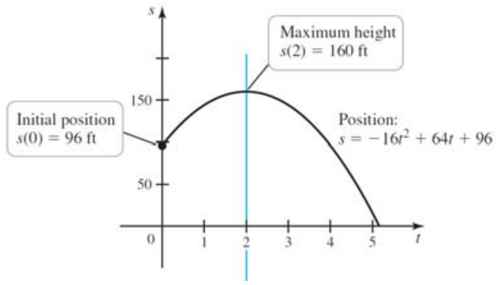
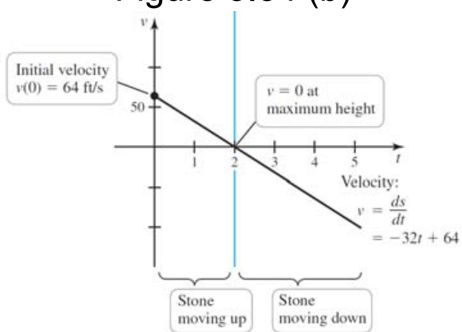


Figure 3.34 (b)



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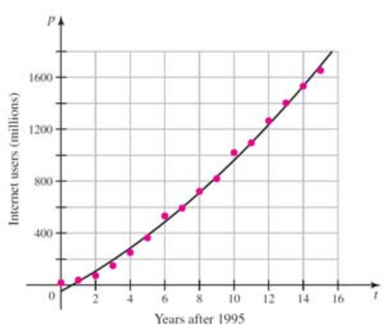

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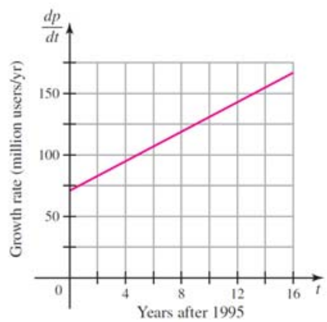

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Figure 3.37

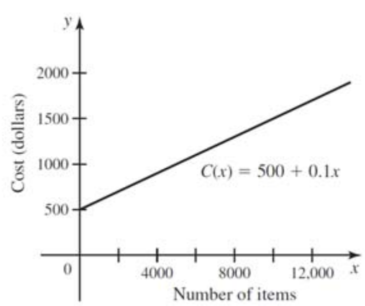

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Figure 3.38

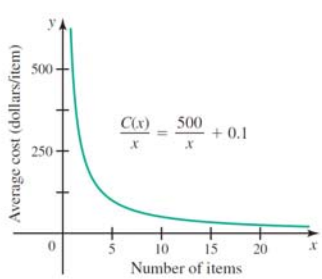

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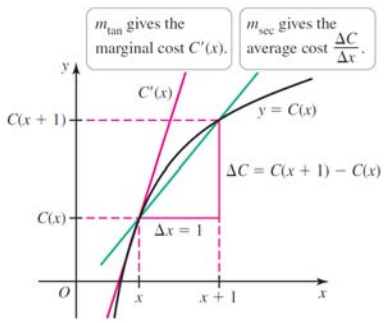
Figure 3.39



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DEFINITION Average and Marginal Cost

The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process. The **average cost** to produce x items is $\overline{C}(x) = C(x)/x$. The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items.

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Figure 3.40

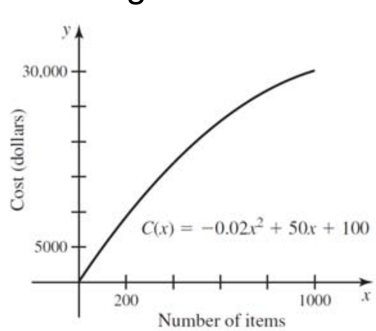

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Figure 3.41 (a)

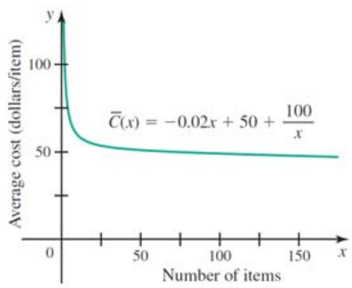

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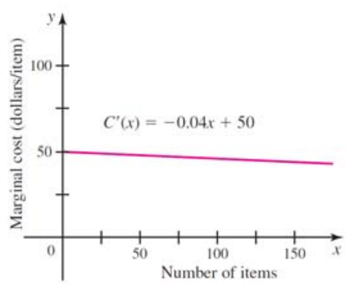
Figure 3.41 (b)



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3.6

The Chain Rule

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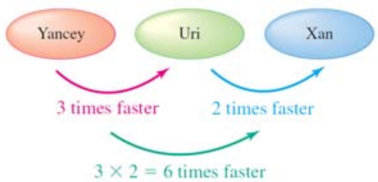
Figure 3.42



THEOREM 3.12 The Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{Version 1}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{Version 2}$$

PROCEDURE Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f and the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:
$$y = f(\underbrace{g(x)}_{u}) \Rightarrow y = f(u).$$
3. Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$.
4. Replace u by $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$.

Table 3.2

x	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	3
2	7	1	4

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THEOREM 3.13 Chain Rule for Powers
If g is differentiable for all x in its domain and n is an integer, then

$$\frac{d}{dx}[(g(x))^n] = n(g(x))^{n-1}g'(x).$$

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3.7

Implicit Differentiation

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Figure 3.43 (a & b)

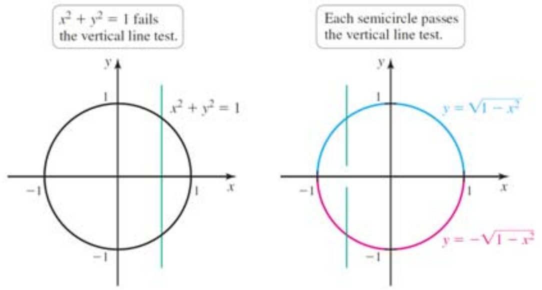


Figure 3.44 (a)

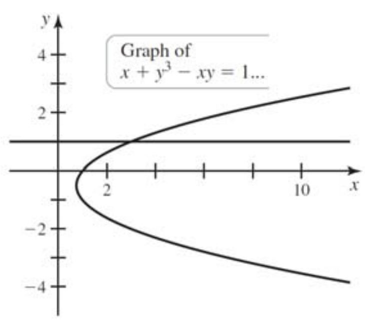


Figure 3.44 (b)

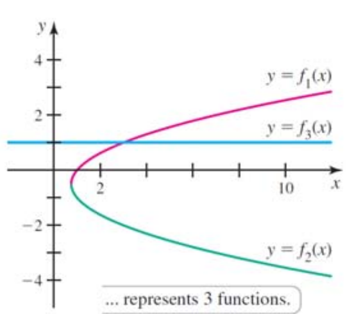


Figure 3.45

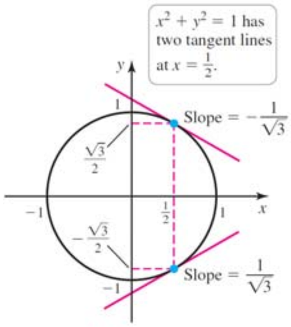
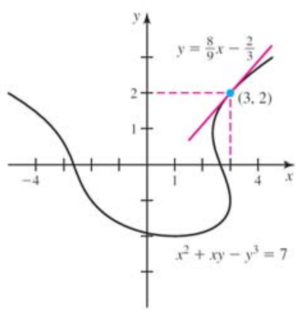


Figure 3.46

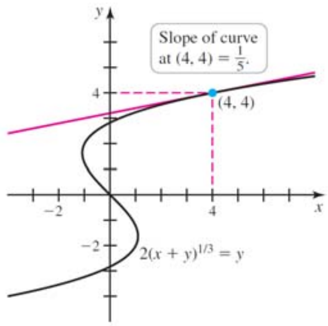


THEOREM 3.14 Power Rule for Rational Exponents
Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{p/q}) = \frac{p}{q}x^{p/q-1},$$

provided $x > 0$ when q is even.

Figure 3.47



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3.8

Related Rates

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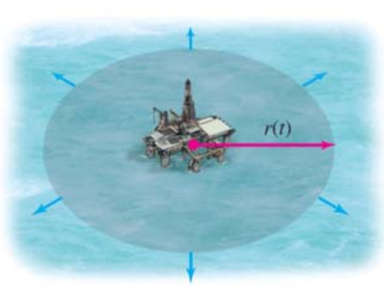
Figure 3.48



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PROCEDURE Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

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Figure 3.49

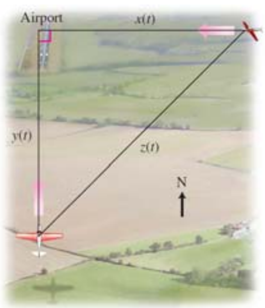

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Figure 3.50

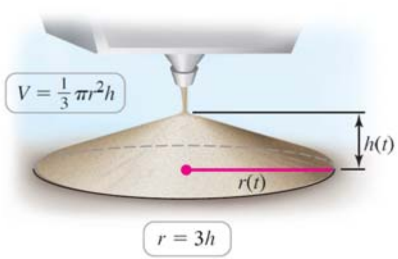

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Figure 3.51

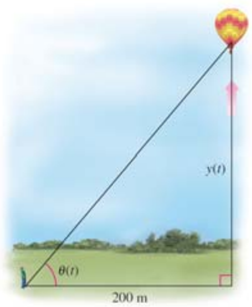

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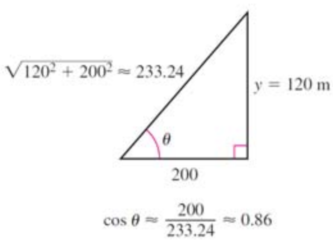
Figure 3.52



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