Chapter 4

Calculus
for Scientists and Engineers

BRIGGS   COCHRAN   GILLETT

Applications of
the Derivative
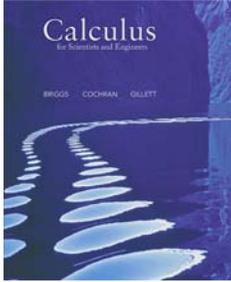
4.1

Maxima and Minima

1

Figure 4.1

High points   Low points

A hike along the x-axis
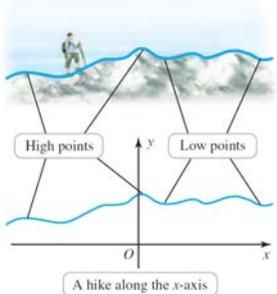
DEFINITION **Absolute Maximum and Minimum**

Let f be defined on an interval I containing c. If $f(c) \geq f(x)$ for every x in I, then f has an **absolute maximum** value of $f(c)$ on I at c. If $f(c) \leq f(x)$ for every x in I, then f has an **absolute minimum** value of $f(c)$ on I at c.
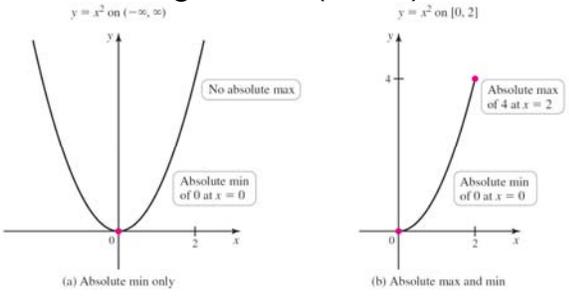
Figure 4.2 (a & b)

$y = x^2$ on $(-\infty, \infty)$

No absolute max

Absolute min of 0 at $x = 0$

(a) Absolute min only

$y = x^2$ on $[0, 2]$

Absolute max of 4 at $x = 2$

Absolute min of 0 at $x = 0$

(b) Absolute max and min

2

Figure 4.2 (c & d)

$y = x^2$ on $(0, 2]$

Absolute max of 4 at $x = 2$

No absolute min

(c) Absolute max only

$y = x^2$ on $(0, 2)$

No absolute max

No absolute min

(d) No max or min

Figure 4.3

$$y = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

No absolute maximum value

Absolute minimum value of 0 at $x = 0$ and $x = 1$

No absolute maximum value

Absolute minimum

$y = f(x)$

THEOREM 4.1   Extreme Value Theorem
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

3

Figure 4.4
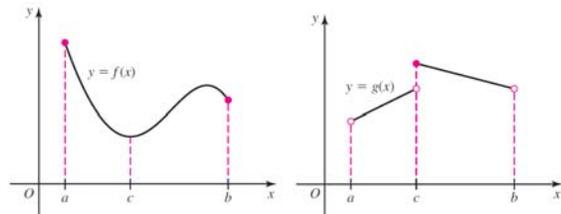

$y = f(x)$

$y = g(x)$

Figure 4.5
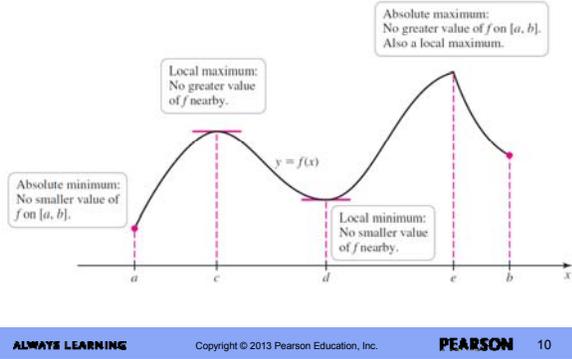
DEFINITION   Local Maximum and Minimum Values

Suppose I is an interval on which f is defined and c is an interior point of I. If $f(c) \geq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local minimum** value of f.
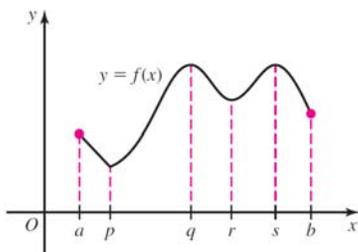
4

Figure 4.6

Figure 4.7

Local maximum at c

$y = f(x)$

Slopes of secant lines ≥ 0

Slopes of secant lines ≤ 0

c x
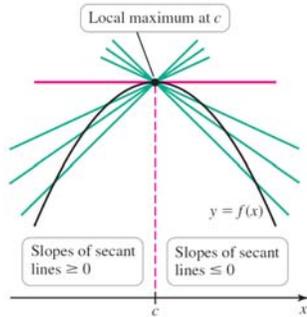
THEOREM 4.2 Local Extreme Point Theorem

If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

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DEFINITION Critical Point

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Figure 4.8 (a & b)
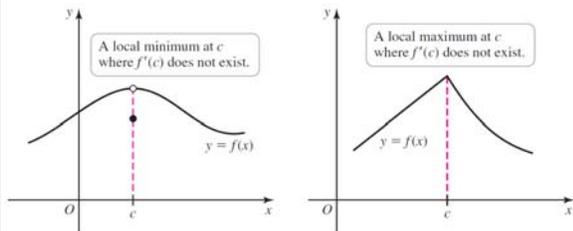
A local minimum at c where $f'(c)$ does not exist.

$y = f(x)$

A local maximum at c where $f'(c)$ does not exist.

$y = f(x)$

Figure 4.9 (a)

$y = f(x)$

$f'(c) = 0$, but no local max/min at c.

6

Figure 4.9 (b)

$y = f(x)$

$f'(c)$ does not exist, but no local max/min at c.

Figure 4.10

$$f(x) = \frac{x}{x^2 + 1}$$

Local (and absolute) maximum.

$\left(1, \frac{1}{2}\right)$

$\left(-1, -\frac{1}{2}\right)$

Local (and absolute) minimum.
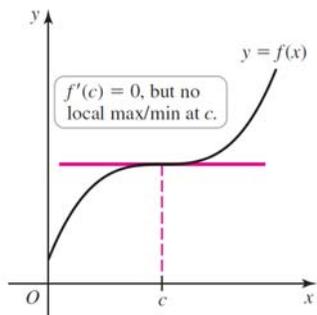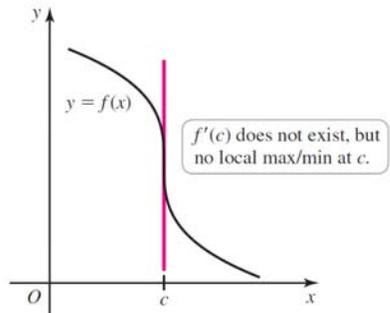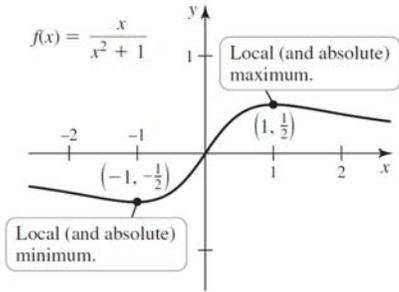
PROCEDURE   **Locating Absolute Maximum and Minimum Values**

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.

2. Evaluate f at the critical points and at the endpoints of $[a, b]$.

3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.
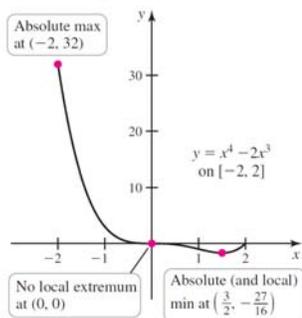
7

Figure 4.11

Absolute max at $(-2, 32)$

$y = x^4 - 2x^3$
on $[-2, 2]$

No local extremum at $(0, 0)$

Absolute (and local) min at $\left(\frac{3}{2}, -\frac{27}{16}\right)$

Figure 4.12

Absolute max at $(-1, 3)$

$y = x^{2/3}(2 - x)$ on $[-1, 2]$

Local max at $(0.8, 1.03)$

Absolute min at $(2, 0)$

Absolute (and local) min at $(0, 0)$
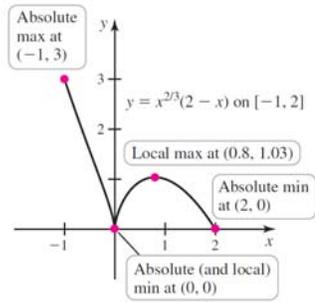
4.2

What Derivatives Tell Us

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DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.
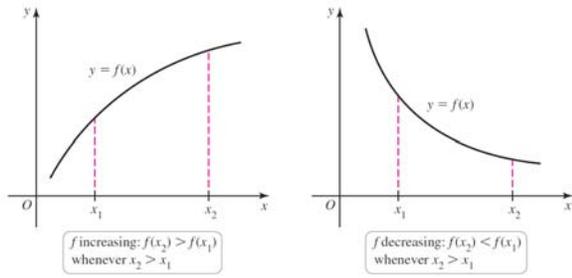
Figure 4.13

$y = f(x)$

f increasing: $f(x_2) > f(x_1)$ whenever $x_2 > x_1$

$y = f(x)$

f decreasing: $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Figure 4.14

Tangent lines have positive slopes ...

$y = f(x)$

... f is increasing.

Tangent lines have negative slopes ...

$y = f(x)$
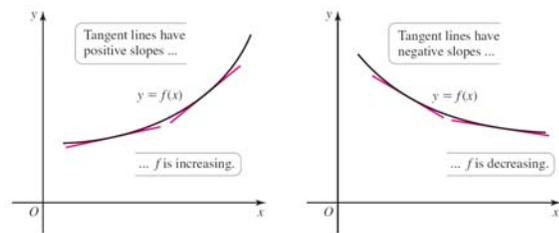
... f is decreasing.

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THEOREM 4.3 Test for Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Figure 4.15

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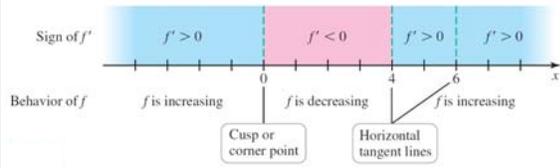**Figure 4.16**

Figure 4.17

Figure 4.18 (a)

f has a local minimum at c.

$y = f(x)$

$f'(x) < 0$ $f'(x) > 0$

Figure 4.18 (b)

f has a local maximum at c.

$y = f(x)$

$f'(x) > 0$ $f'(x) < 0$

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Figure 4.19

Absolute max
f' undefined

Local max
$f' = 0$

No extremum
$f' = 0$

No extremum
$f' = 0$

$y = f(x)$

$f' < 0$

$f' > 0$ $f' < 0$ $f' > 0$ $f' < 0$

Absolute min

Local min
$f' = 0$

$f' > 0$

a c_1 c_2 c_3 c_4 c_5 b
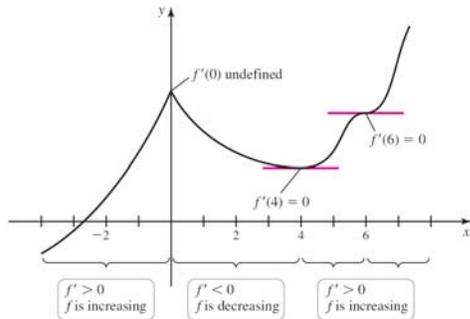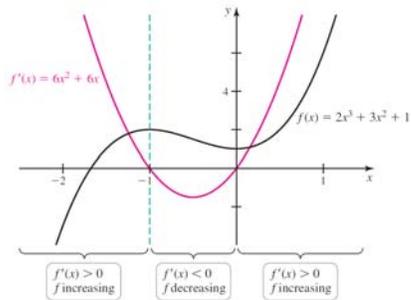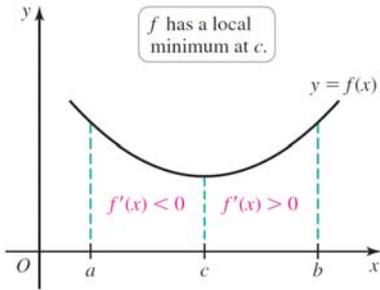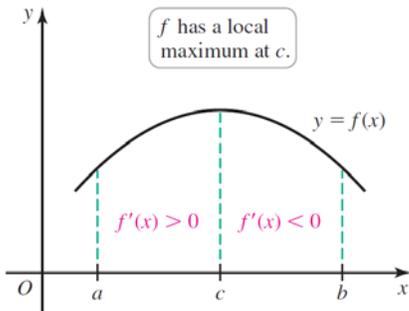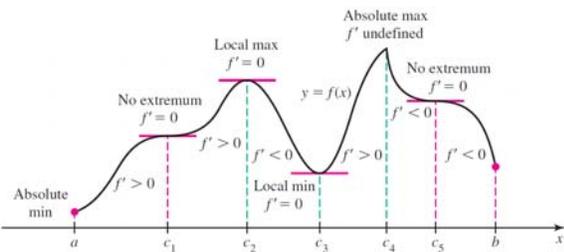
THEOREM 4.4   First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.

- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.

- If f' does not change sign at c (from positive to negative or vice versa), then f has no local extreme value at c.

Figure 4.20

Sign of
$f'(x) = 12(x + 1)(x - 1)^2$

$f' < 0$ $f' > 0$ $f' > 0$

Behavior of f Decreasing   Increasing   Increasing

12

Figure 4.21

$f' < 0$, f decreasing $f' > 0$, f increasing $f' > 0$, f increasing

$y = f(x)$

Local min at $x = -1$

f increases through (1, 6); no local extreme point

$y = f'(x)$

Figure 4.22

Sign of $f'(x) = \dfrac{4-5x}{3x^{1/3}}$

f' does not exist at $x = 0$

$f' = 0$ at $x = \dfrac{4}{5}$

$\dfrac{\text{pos}}{\text{neg}}\ f' < 0$ $\dfrac{\text{pos}}{\text{pos}}\ f' > 0$ $\dfrac{\text{neg}}{\text{pos}}\ f' < 0$

Behavior of f Decreasing   Increasing   Decreasing

Figure 4.23

$f(x) = x^{2/3}(2-x)$

Local max at $x = \dfrac{4}{5}$

Cusp and local min at $x = 0$

$y = f'(x)$

$f' < 0$ f decreasing $f' > 0$ f increasing $f' < 0$ f decreasing

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THEOREM 4.5   One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains one local extremum at c.

• If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
• If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.

Figure 4.24

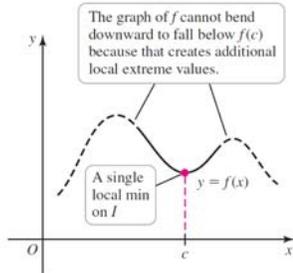The graph of f cannot bend downward to fall below $f(c)$ because that creates additional local extreme values.

A single local min on I

$y = f(x)$

Figure 4.25

$y = \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - 9x + 2$

$\left(3, -\frac{73}{4}\right)$

Absolute minimum at $x = 3$.

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Figure 4.26

$y = x^3$

Concave up

f' decreases and slopes of tangent lines decrease as x increases

f' increases and slopes of tangent lines increase as x increases
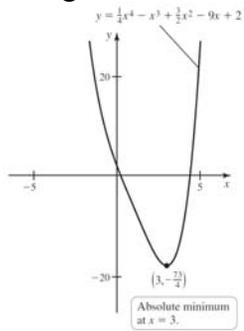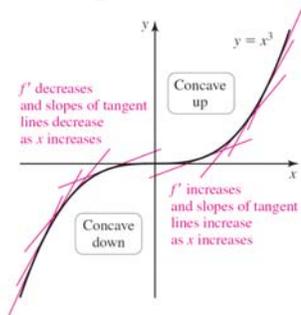
Concave down

DEFINITION Concavity and Inflection Point

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

THEOREM 4.6   Test for Concavity

Suppose that f'' exists on an interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
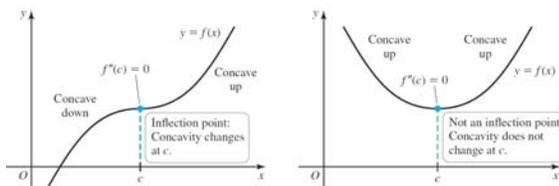- If c is a point of I at which f'' changes sign at c, then f has an inflection point at c.

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Figure 4.27 (a & b)

Figure 4.27 (c & d)
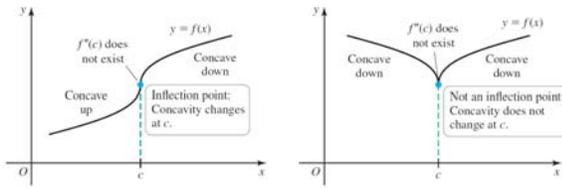
Figure 4.28 (a)
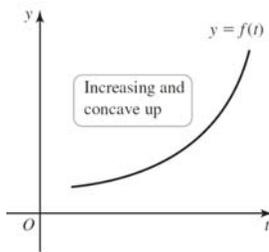
16

Figure 4.28 (b)

Figure 4.29

Sign of
$f''(x) = 12(x-1)(3x+1)$

| $f'' > 0$ | $f'' < 0$ | $f'' > 0$ |

$-\frac{1}{3}$ 1

Behavior of f Concave up   Concave down   Concave up

Inflection points
at $x = -\frac{1}{3}, 1$

 PEARSON 49

Figure 4.30

$y = f(x)$ y $y = f''(x)$

10

-2 -1 2 x

Inflection point at $x = 1$

Inflection point at $x = -\frac{1}{3}$

| $f'' > 0$ f concave up | $f'' < 0$ f concave down | $f'' > 0$ f concave up |

 PEARSON 50

17

Figure 4.31 (a)

y

$f'(c) = 0$ and $f''(c) > 0$.
Local minimum at c

$y = f(x)$

O c x

 PEARSON 51

Figure 4.31 (b)

$f'(c) = 0$ and $f''(c) < 0$.
Local maximum at c

$y = f(x)$
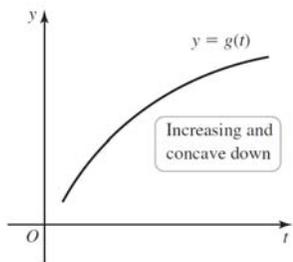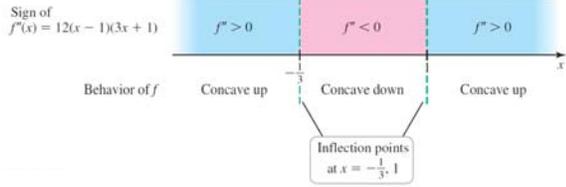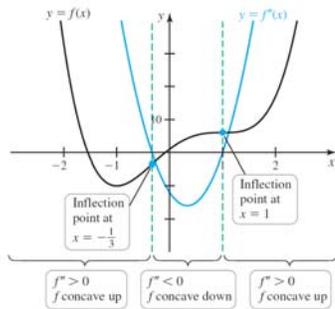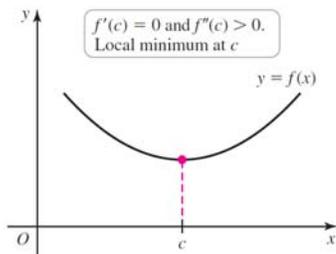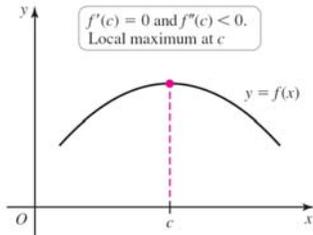
THEOREM 4.7   **Second Derivative Test for Local Extrema**
Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

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Figure 4.32
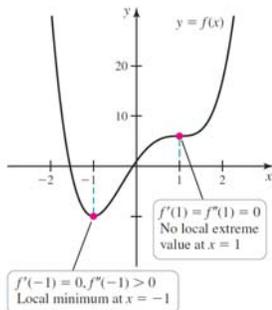
$y = f(x)$

$f'(1) = f''(1) = 0$
No local extreme value at $x = 1$

$f'(-1) = 0, f''(-1) > 0$
Local minimum at $x = -1$

Figure 4.33

$f'\left(-\frac{\pi}{2}\right)=0, f''\left(-\frac{\pi}{2}\right)<0$
Local maximum at $x=-\frac{\pi}{2}$

$f'\left(\frac{\pi}{2}\right)=0, f''\left(\frac{\pi}{2}\right)<0$
Local maximum at $x=\frac{\pi}{2}$

$y=f(x)=\sin^2 x$

$f'(-\pi)=0, f''(-\pi)>0$
Local minimum at $x=-\pi$

$f'(0)=0, f''(0)>0$
Local minimum at $x=0$

$f'(\pi)=0, f''(\pi)>0$
Local minimum at $x=\pi$
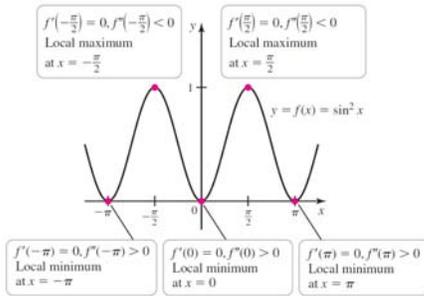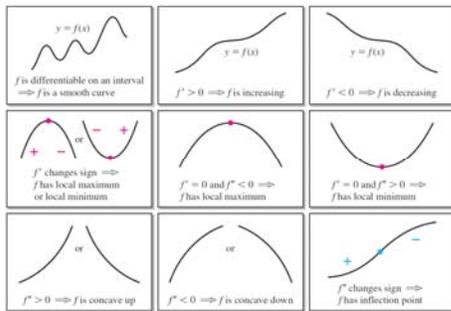
Figure 4.34

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4.3

Graphing Functions

Figure 4.35 (a)
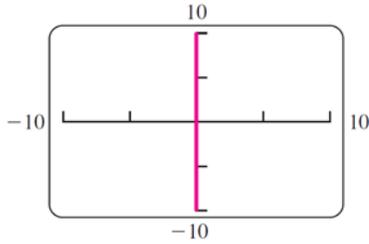
Figure 4.35 (b)
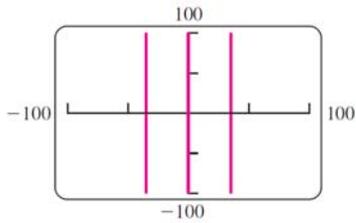
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Graphing Guidelines for $y = f(x)$

1. **Identify the domain or interval of interest.** On what interval should the function be graphed? It may be the domain of the function or some subset of the domain.

2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function *even* $(f(-x) = f(x))$, *odd* $(f(-x) = -f(x))$, or neither?

3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.

4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.

continued...

5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.

6. **Identify extreme values and inflection points.** Use either the First or the Second Derivative Test to classify the critical points. Both *x*- and *y*-coordinates of maxima, minima, and inflection points are needed for graphing.

7. **Locate vertical/horizontal asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior.

8. **Find the intercepts.** The *y*-intercept of the graph is found by setting $x = 0$. The *x*-intercepts are the real zeros (or roots) of a function: those values of *x* that satisfy $f(x) = 0$.

9. **Choose an appropriate graphing window and make a graph.** Use the results of the above steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

Figure 4.36

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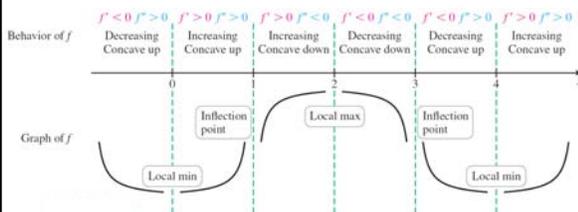## Figure 4.37

Figure 4.38

Critical points

Sign of
$f'(x) = (x - 20)(x + 20)$

$f' > 0$ $f' < 0$ $f' > 0$

-20 20 x

Behavior of f Increasing   Decreasing   Increasing

Figure 4.39

Inflection point

Sign of
$f''(x) = 2x$

$f'' < 0$ $f'' > 0$

0 x

Behavior of f Concave down   Concave up

22

Figure 4.40

$f' > 0\ f'' < 0$ $f' < 0\ f'' < 0$ $f' < 0\ f'' > 0$ $f' > 0\ f'' > 0$

Behavior of f Increasing   Decreasing   Decreasing   Increasing
Concave down   Concave down   Concave up   Concave up

-20 0 20 x

Local max

Graph of f

Inflection point   Local min

Figure 4.41

Local maximum

$(-20, 5333\frac{1}{3})$

$y = \frac{x^3}{3} - 400x$

Inflection point

-20 0 20 x

-6000 $(20, -5333\frac{1}{3})$

Local minimum

Table 4.1

	$20x$	$x^2 + 3$	$(x^2 - 1)^3$	Sign of f''
$(-\infty, -1)$	$-$	$+$	$+$	$-$
$(-1, 0)$	$-$	$+$	$-$	$+$
$(0, 1)$	$+$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

23

Figure 4.42

Behavior of f	$f' > 0\; f'' < 0$ Increasing Concave down	$f' < 0\; f'' < 0$ Decreasing Concave down	$f' < 0\; f'' > 0$ Decreasing Concave up	$f' < 0\; f'' < 0$ Decreasing Concave down	$f' < 0\; f'' > 0$ Decreasing Concave up	$f' > 0\; f'' > 0$ Increasing Concave up

$-\sqrt{3}$ -1 0 $\sqrt{3}$ x

Local max

Graph of f

Local min

Inflection point

Asymptote   Asymptote

Figure 4.43

Table 4.2

24

Figure 4.44

Figure 4.45

$$y = \frac{x^{2/3}(9x^2 - 8x - 16)}{8}$$

Local maximum

(0, 0)

$\left(-\frac{4}{9}, -0.78\right)$

Local minimum

$\left(1, -\frac{15}{8}\right)$

Absolute minimum

PEARSON

73

4.4

Optimization Problems

25

PEARSON

74

Table 4.3

x	y	$x + y$	$P = xy$
1	19	20	19
5.5	14.5	20	79.75
9	11	20	99
13	7	20	91
18	2	20	36

PEARSON

75

Figure 4.46

Maximum product
$P = 100$

Objective function P

$(0, 20)$ $x + y = 20$

$(10, 10)$

(x, y)

$(20, 0)$

Figure 4.47

Barn

x x x x

y

Constraint: $4x + y = 400$

26

Figure 4.48

Optimal solution:
$(50, 10{,}000)$
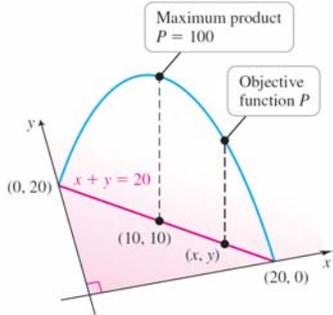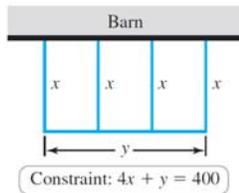
Objective function:
$A(x) = 400x - 4x^2$
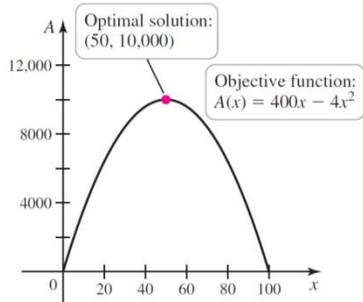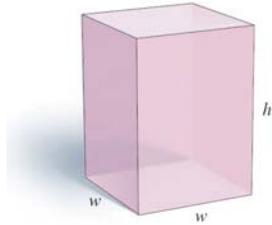
Figure 4.49

Objective function: $V = w^2h$
Constraint: $2w + h = 64$

Figure 4.50

Optimal solution at $w = \frac{64}{3}$

Objective function:
$V = 64w^2 - 2w^3$
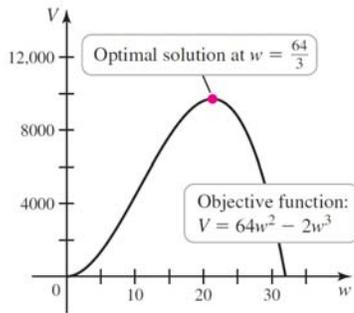
27

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.

2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.

3. Identify the constraint(s). Write them in terms of the variables of the problem.

4. Use the constraint(s) to eliminate all but one independent variable of the objective function.

5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.

6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.
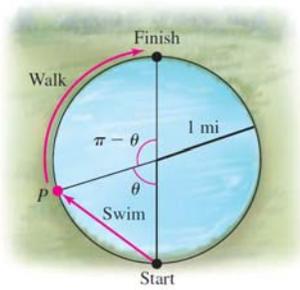
Figure 4.51

Figure 4.52

Chord length $= 2r \sin\left(\frac{\theta}{2}\right)$

Arc length $= r\theta$

$r \sin\left(\frac{\theta}{2}\right)$

28

Figure 4.53

Max. travel time ≈ 1.23

Min. travel time $= 1$

$\left(0, \frac{\pi}{3}\right)$ $(\pi, 1)$
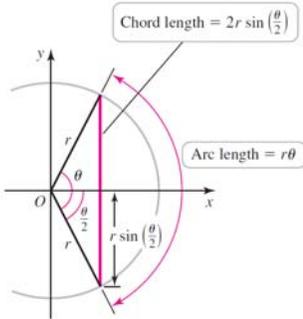
Objective function:
$T(\theta) = \sin\frac{\theta}{2} + \frac{\pi - \theta}{3}$
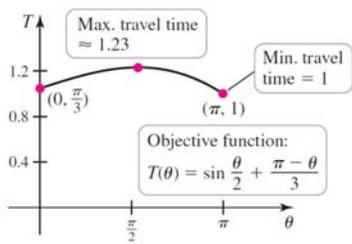
Figure 4.54 (a)

Figure 4.54 (b)
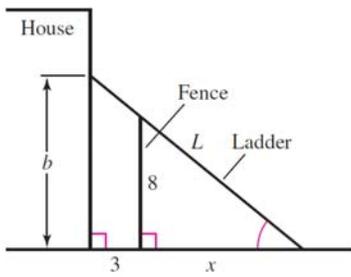
29

4.5

Linear Approximation
and Differentials

Figure 4.55 (a & b)

Figure 4.56

Tangent line at $(a, f(a))$:
$L(x) = f(a) + f'(a)(x - a)$

$y = f(x)$
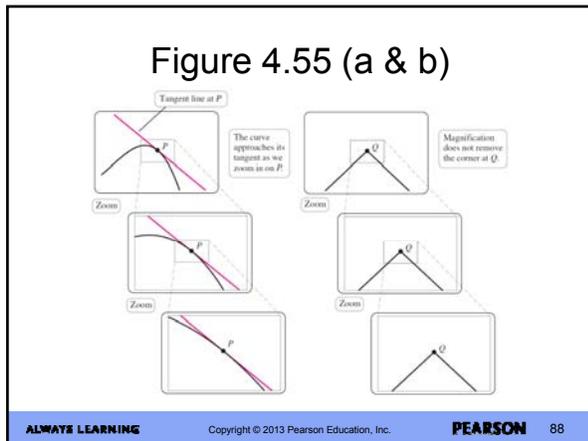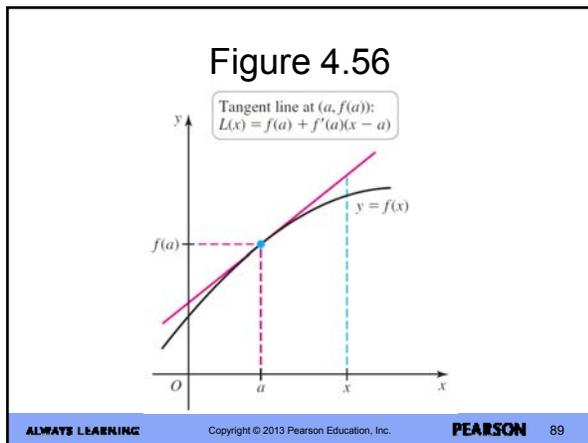
30

DEFINITION **Linear Approximation to f at a**

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Figure 4.57

Linear approximation:
$L(x) = \frac{1}{2}(x + 1)$

$(1, 1)$

$f(x) = \sqrt{x}$

$L(1.1) = 1.05$
$\sqrt{1.1} = 1.0488\ldots$

The error in the approximation decreases as $x \to 1$.
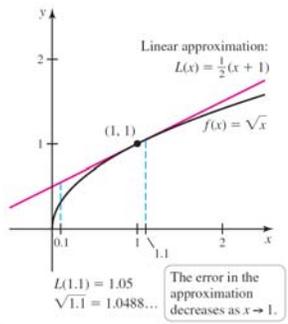
Table 4.4

x	$L(x)$	Exact $\sqrt{x}$	Percent error
1.2	1.1	1.0954 ...	0.4%
1.1	1.05	1.0488 ...	0.1%
1.01	1.005	1.0049 ...	0.001%
1.001	1.0005	1.0005 ...	0.00001%

31

Figure 4.58

Linear approximation:
$L(x) = x$

$f(x) = \sin x$
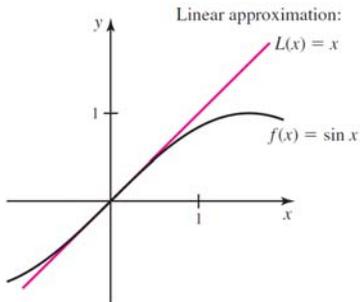
Figure 4.59
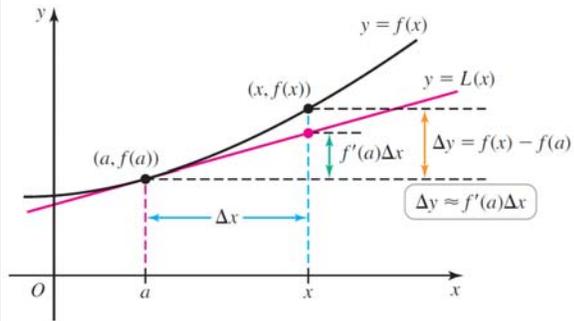
Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a)\, \Delta x,$$

where $a + \Delta x$ is in I.

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SUMMARY   Uses of Linear Approximation

- To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

- To approximate the change Δy in the dependent variable when x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\, \Delta x.$$

DEFINITION **Differentials**

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is,

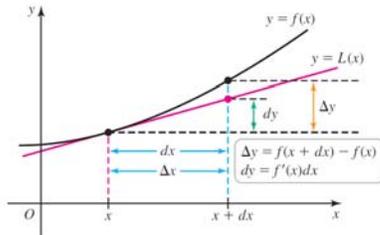$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

Figure 4.60

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

33

4.6

Mean Value Theorem

Figure 4.61

$f'(c) = 0$

$y = f(x)$
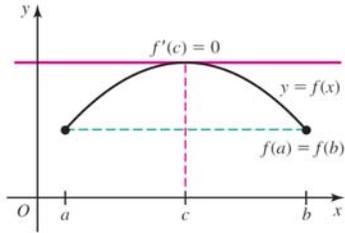
$f(a) = f(b)$

THEOREM 4.8   Rolle's Theorem

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.
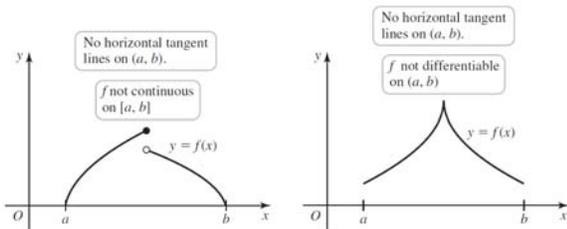
34

Figure 4.62

No horizontal tangent lines on (a, b).

f not continuous on $[a, b]$

$y = f(x)$

No horizontal tangent lines on (a, b).

f not differentiable on (a, b)

$y = f(x)$

Figure 4.63

$y = x^3 - 7x^2 + 10x$

3.79

0

0.88

3

5

$f'(c) = 0$
at two points
in $(0, 5)$.

Figure 4.64

These lines are parallel and
their slopes are equal, that is...

Secant line: slope $= \dfrac{f(b) - f(a)}{b - a}$

Tangent line: slope $= f'(c)$

$y = f(x)$

O a c b x
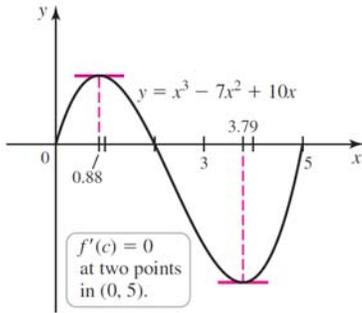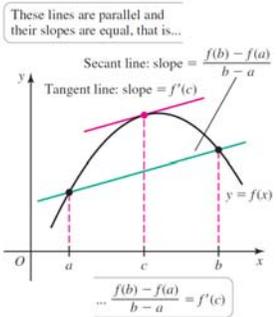
$\dots \dfrac{f(b) - f(a)}{b - a} = f'(c)$

35

THEOREM 4.9   Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Figure 4.65
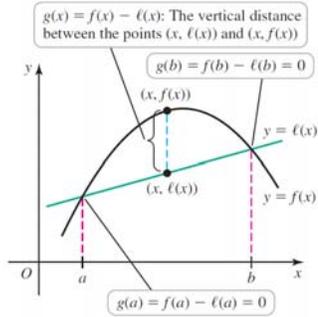
$g(x) = f(x) - \ell(x)$: The vertical distance between the points $(x, \ell(x))$ and $(x, f(x))$

$g(b) = f(b) - \ell(b) = 0$

$(x, f(x))$

$y = \ell(x)$

$(x, \ell(x))$

$y = f(x)$

$g(a) = f(a) - \ell(a) = 0$

Figure 4.66

$(2900\text{ m}, 7.6°C)$

We don't know the temperature function between the points.

Average rate of change $= -8.1°C/km$

The MVT guarantees at least one point where the slope of the tangent line is equal to the slope of the secant line.

$(5600\text{ m}, -14.3°C)$

Temperature (°C)

Altitude (m)

36

Figure 4.67

Slope of secant line $= 5$

Slope $= 5$

Slope $= 5$

$y = 2x^3 - 3x + 1$
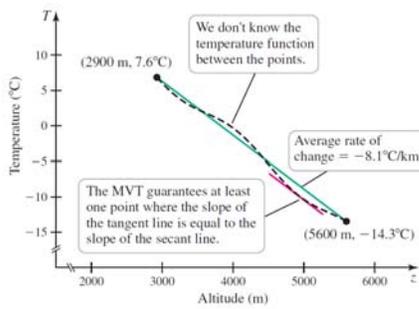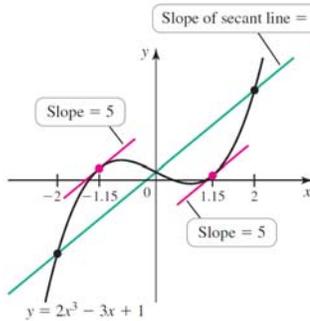
THEOREM 4.10 Zero Derivative Implies Constant Function

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

THEOREM 4.11 Functions with Equal Derivatives Differ by a Constant

If two functions have the property that $f'(x) = g'(x)$, for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant; that is, f and g differ by a constant.

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THEOREM 4.12 Intervals of Increase and Decrease

Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

4.7

L'Hôpital's Rule

THEOREM 4.13 **L'Hôpital's Rule**

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

38

Figure 4.68

$$\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} = \frac{4}{2} = 2$$
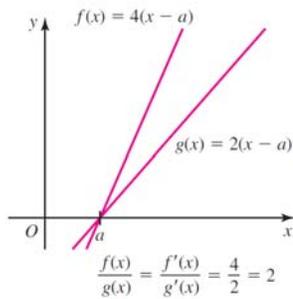
Figure 4.69
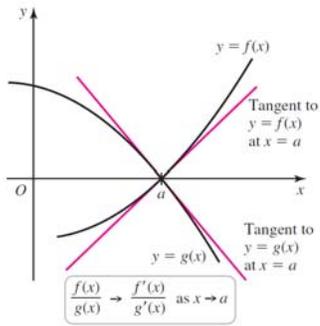
THEOREM 4.14   L'Hôpital's Rule (∞/∞)

Suppose that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

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4.8

Newton's Method

Figure 4.70 (a)
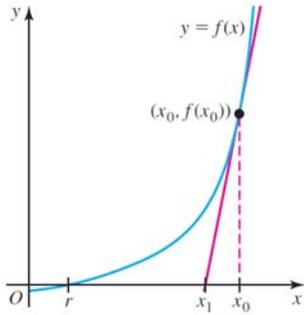
Figure 4.70 (b)
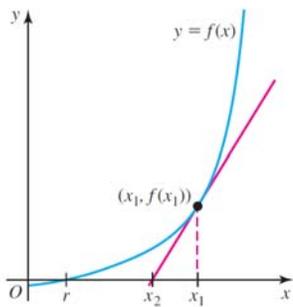
40

Figure 4.70 (c)
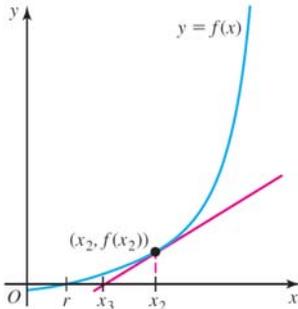
Figure 4.71
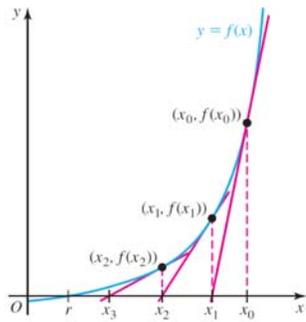
PROCEDURE Newton's Method for Approximating Roots of $f(x) = 0$

1. Choose an initial approximation x_0 as close to a root as possible.
2. For $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided $f'(x_n) \neq 0$.

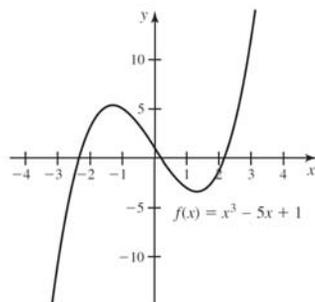3. End the calculations when a termination condition is met.

41

Figure 4.72

$f(x) = x^3 - 5x + 1$

Table 4.5

k	x_k	x_k	x_k
0	-3	1	4
1	-2.500000	-0.500000	2.953488
2	-2.345455	0.294118	2.386813
3	-2.330203	0.200215	2.166534
4	-2.330059	0.201639	2.129453
5	-2.330059	0.201640	2.128420
6	-2.330059	0.201640	2.128419
7	-2.330059	0.201640	2.128419

Figure 4.73 (1 of 2)
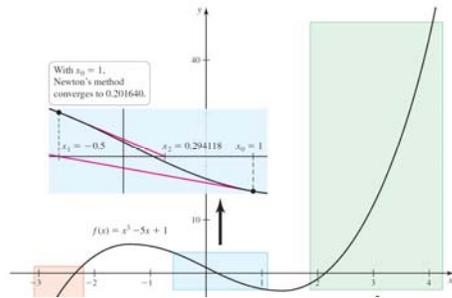
42

Figure 4.73 (2 of 2)

Figure 4.74

$y = x$

$y = \cos x$

$y = \cos x - x$
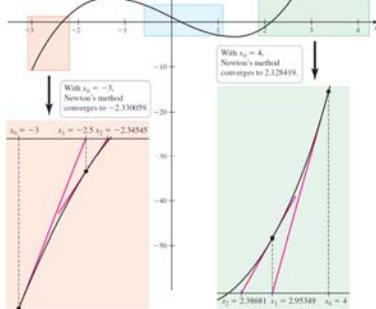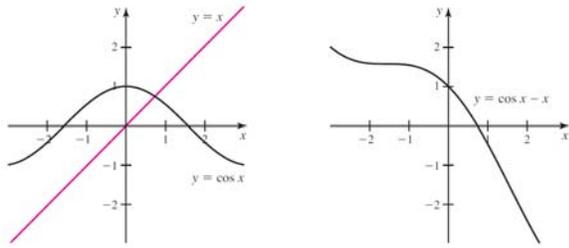
Table 4.6

k	x_k	Residual
0	0.5	0.377583
1	0.755222	−0.0271033
2	0.739142	−0.0000946154
3	0.739085	-1.18098×10^{-9}
4	0.739085	0
5	0.739085	0
6	0.739085	0
7	0.739085	0
8	0.739085	0
9	0.739085	0
10	0.739085	0

43

Figure 4.75

$y = x^5 - x^4 + x^2 - x$
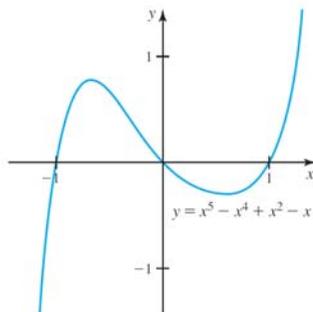
Table 4.7

k	x_k	x_k
0	0.200000	2.500000
1	0.499372	1.623915
2	0.550979	2.062202
3	0.553568	2.121018
4	0.553574	2.124360
5	0.553574	2.124371
6	0.553574	2.124371

Table 4.8

k	x_k	x_k	x_k
0	1	0.15	1.1
1	-1	0.0699375	-1.4465
2	1	0.0344556	3.81665
3	-1	0.0171665	-81.4865
4	1	0.00857564	8.11572×10^5
5	-1	0.00428687	-8.01692×10^{17}

44

Figure 4.76

$f(x) = \dfrac{8x^2}{3x^2 + 1}$

Figure 4.77

$$f(x) = \frac{8x^2}{3x^2 + 1}$$

$x_1 = 0.06994 \quad x_0 = 0.15$

$x_3 = 0.01717 \quad x_2 = 0.3446$

Figure 4.78

$$f(x) = \frac{8x^2}{3x^2 + 1}$$

$x_1 = -1.4465 \quad x_0 = 1.1 \quad x_2 = 3.81665$
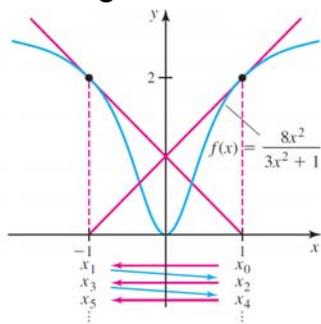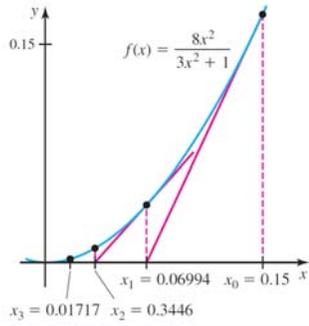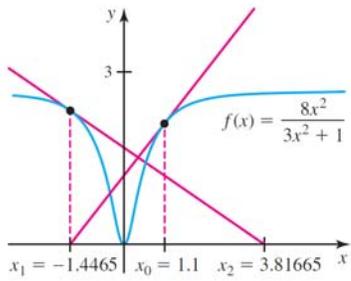
45

4.9

Antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for all x in I.

THEOREM 4.15 The Family of Antiderivatives

Let F be any antiderivative of f on an interval I. Then *all* the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

46

Figure 4.79

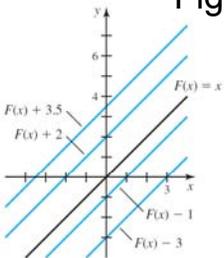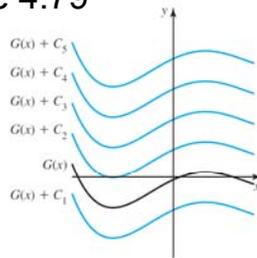Several antiderivatives of $f(x) = 1$ from the family $F(x) + C = x + C$

If G is any antiderivative of g, the graphs of the antiderivatives $G + C$ are vertical translations of one another.

THEOREM 4.16 Power Rule for Indefinite Integrals

$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

THEOREM 4.17 Constant Multiple and Sum Rules

Constant Multiple Rule: $\int cf(x) \, dx = c \int f(x) \, dx$, for real numbers c

Sum Rule: $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Table 4.9

1. $\dfrac{d}{dx}(\sin ax) = a \cos ax \quad \rightarrow \quad \int \cos ax \, dx = \dfrac{1}{a}\sin ax + C$

2. $\dfrac{d}{dx}(\cos ax) = -a \sin ax \quad \rightarrow \quad \int \sin ax \, dx = -\dfrac{1}{a}\cos ax + C$

3. $\dfrac{d}{dx}(\tan ax) = a \sec^2 ax \quad \rightarrow \quad \int \sec^2 ax \, dx = \dfrac{1}{a}\tan ax + C$

4. $\dfrac{d}{dx}(\cot ax) = -a \csc^2 ax \quad \rightarrow \quad \int \csc^2 ax \, dx = -\dfrac{1}{a}\cot ax + C$

5. $\dfrac{d}{dx}(\sec ax) = a \sec ax \tan ax \quad \rightarrow \quad \int \sec ax \tan ax \, dx = \dfrac{1}{a}\sec ax + C$

6. $\dfrac{d}{dx}(\csc ax) = -a \csc ax \cot ax \quad \rightarrow \quad \int \csc ax \cot ax \, dx = -\dfrac{1}{a}\csc ax + C$
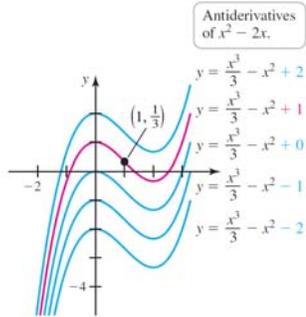
47

Figure 4.80

Antiderivatives of $x^2 - 2x$.

$y = \frac{x^3}{3} - x^2 + 2$

$y = \frac{x^3}{3} - x^2 + 1$

$y = \frac{x^3}{3} - x^2 + 0$

$y = \frac{x^3}{3} - x^2 - 1$

$y = \frac{x^3}{3} - x^2 - 2$

$\left(1, \frac{1}{3}\right)$

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem

$$s'(t) = v(t), \quad s(0) = s_0, \quad \text{where } s_0 \text{ is the initial position.}$$

If the acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), \quad v(0) = v_0, \quad \text{where } v_0 \text{ is the initial velocity.}$$

48

Figure 4.81

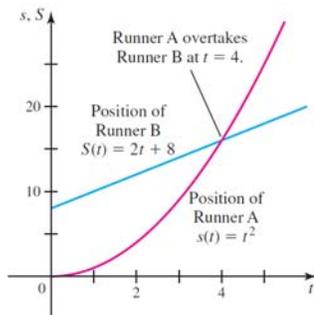

Runner A overtakes Runner B at $t = 4$.

Position of Runner B $S(t) = 2t + 8$

Position of Runner A $s(t) = t^2$

Figure 4.82

Object initially moves upwards.

Maximum height, $s \approx 182$ m.

$s = 100$

Object falls to the river after reaching its maximum height.

Height = 100 m

$s = 0$ River

Figure 4.83

Positive velocity, upward motion

The velocity is 0 at $t \approx 4.1$ seconds, the high point of the trajectory.

$v(t) = -9.8t + 40$

Negative velocity, downward motion

49

Figure 4.84

A maximum height of 182 m is reached at $t \approx 4.1$ s.

$s(t) = -4.9t^2 + 40t + 100$

This is the graph of the position function, not the path of the stone.

182

100

Object strikes the water at $t \approx 10.2$ s.