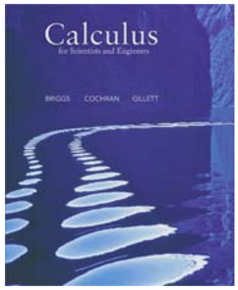
Chapter 5

Integration



5.1

Approximating Areas under Curves

1

Figure 5.1

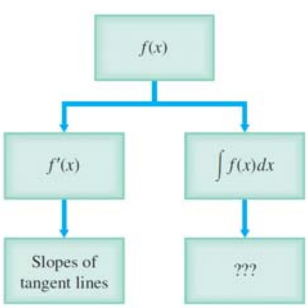


Figure 5.2

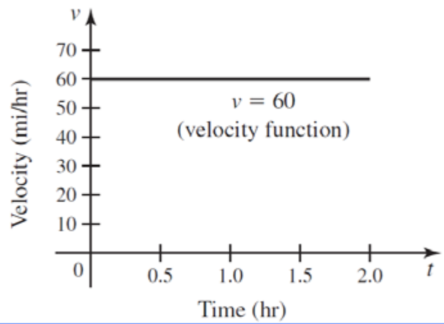


Figure 5.3

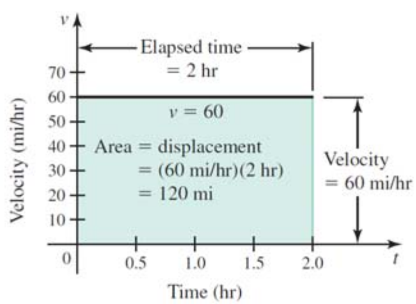


Figure 5.4 (a)

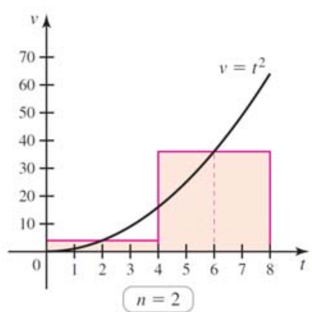


Figure 5.4 (b)

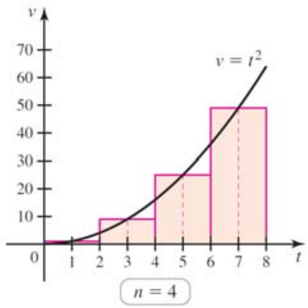


Figure 5.4 (c)

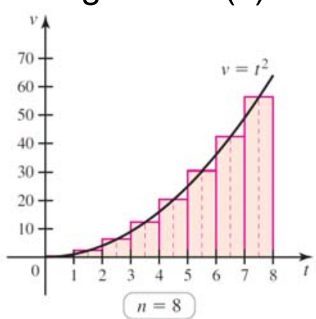


Figure 5.5

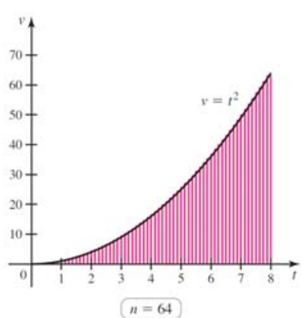


Table 5.1

Approximations to the area under the velocity curve $v = t^2$ on $[0, 8]$

Number of subintervals	Length of each subinterval	Approximate displacement (area under curve)
1	8 s	128.0 m
2	4 s	160.0 m
4	2 s	168.0 m
8	1 s	170.0 m
16	0.5 s	170.5 m
32	0.25 s	170.625 m
64	0.125 s	170.65625 m

Figure 5.6

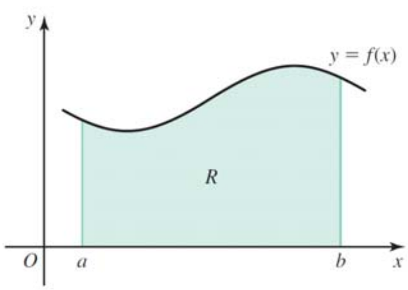
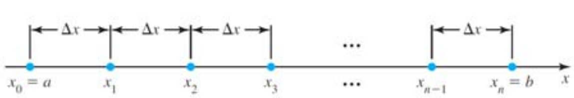


Figure 5.7



DEFINITION Regular Partition

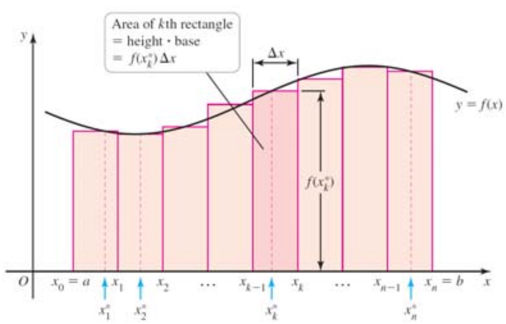
Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \dfrac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \dots, n.$$

Figure 5.8



5

DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \dots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \dots, n$.

Figure 5.9

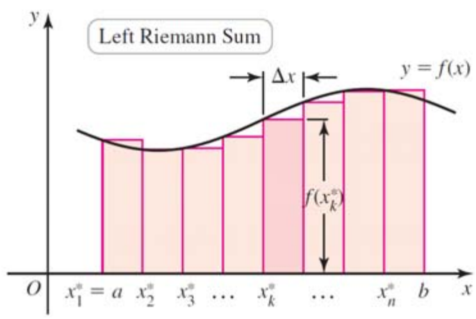


Figure 5.10

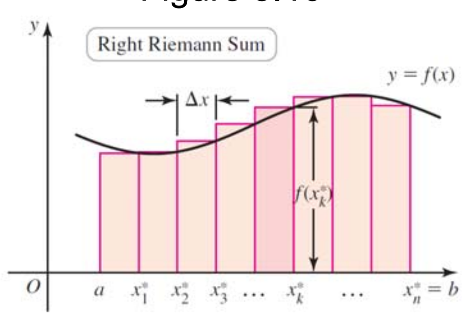


Figure 5.11

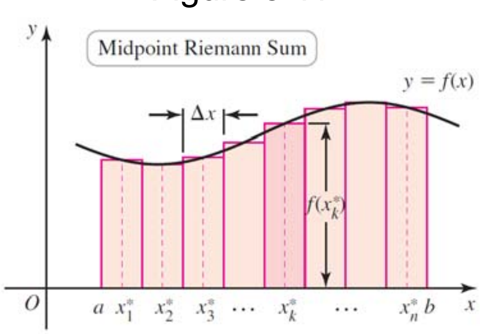


Figure 5.12

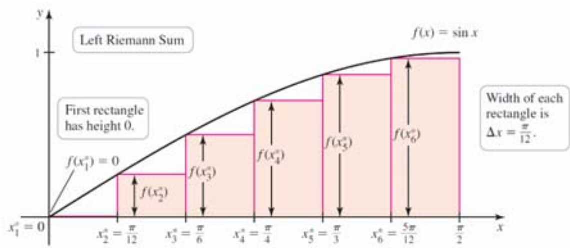


Figure 5.13

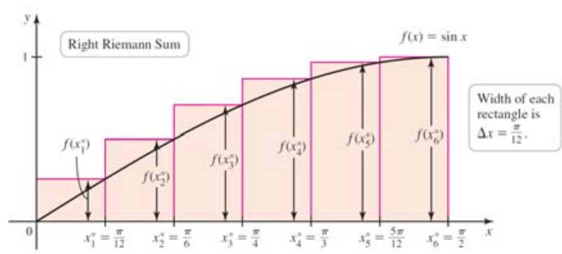


Figure 5.14

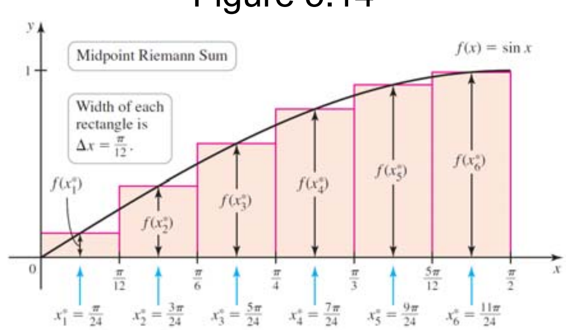


Table 5.2

x	$f(x)$
0	1
0.5	3
1.0	4.5
1.5	5.5
2.0	6.0

THEOREM 5.1 Sums of Positive Integers

Let n be a positive integer.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

8

DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- **left Riemann sum** if $x_k^* = a + (k - 1)\Delta x$
- **right Riemann sum** if $x_k^* = a + k\Delta x$
- **midpoint Riemann sum** if $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, 2, \ldots, n$

Figure 5.15

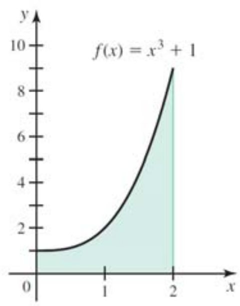


Table 5.3: Left, right, and midpoint Riemann sum approximations

n	L_n	R_n	M_n
20	5.61	6.41	5.995
40	5.8025	6.2025	5.99875
60	5.86778	6.13444	5.99944
80	5.90063	6.10063	5.99969
100	5.9204	6.0804	5.9998
120	5.93361	6.06694	5.99986
140	5.94306	6.05735	5.9999
160	5.95016	6.05016	5.99992
180	5.95568	6.04457	5.99994
200	5.9601	6.0401	5.99995

5.2

Definite Integrals

Figure 5.16

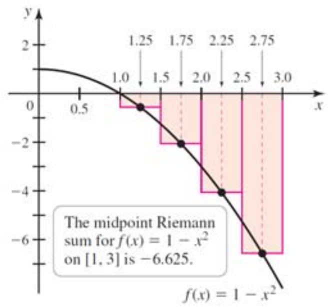


Figure 5.17

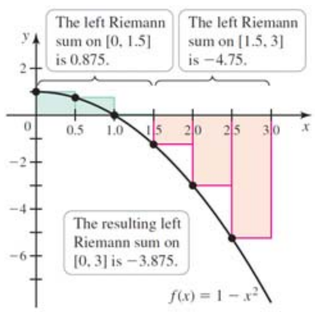


Figure 5.18

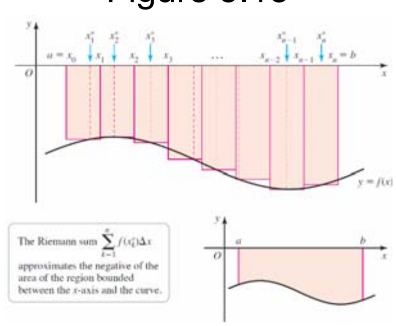
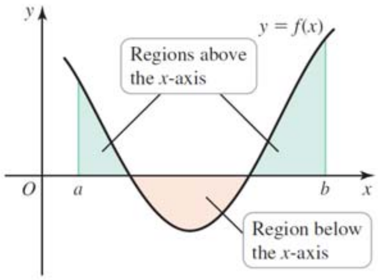


Figure 5.19



DEFINITION Net Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

Figure 5.20 (1 of 2)

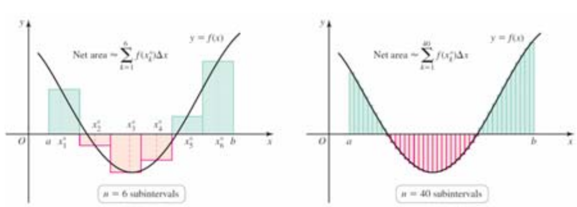
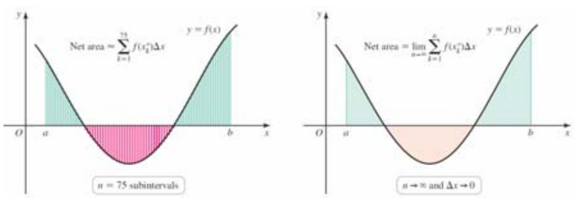


Figure 5.20 (2 of 2)

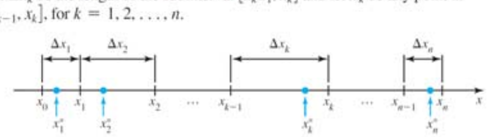


DEFINITION General Riemann Sum

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.



If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Figure 5.21

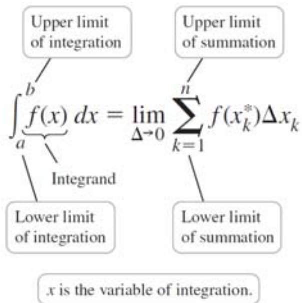
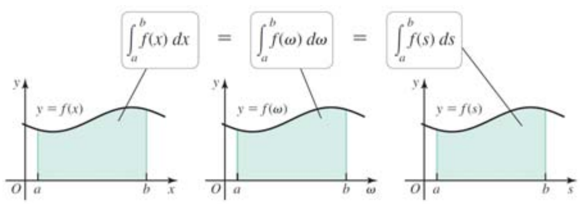


Figure 5.22



THEOREM 5.2 Integrable Functions
If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

Figure 5.23

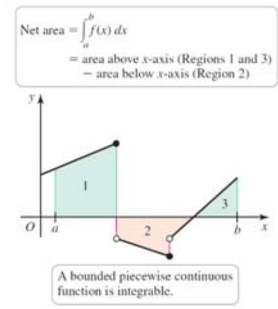


Figure 5.24

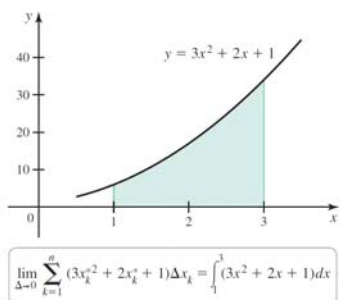


Figure 5.25

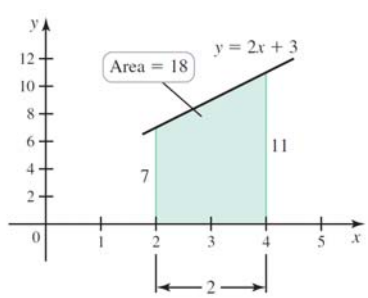


Figure 5.26

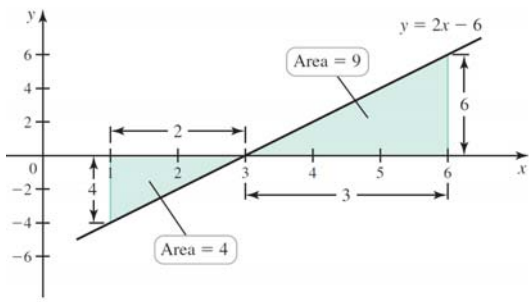
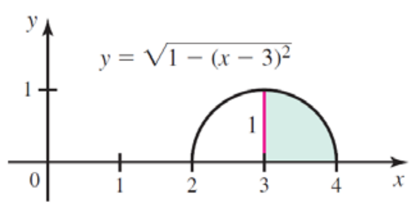
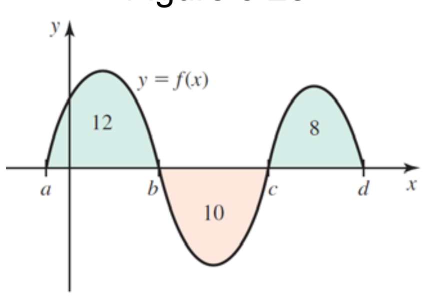


Figure 5.27



Area of shaded region $= \frac{1}{4}\pi(1)^2 = \frac{1}{4}\pi$

Figure 5.28



DEFINITION Reversing Limits and Identical Limits

Suppose f is integrable on $[a, b]$.

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$

Figure 5.29

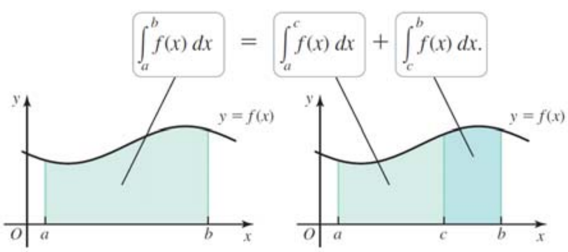


Figure 5.30

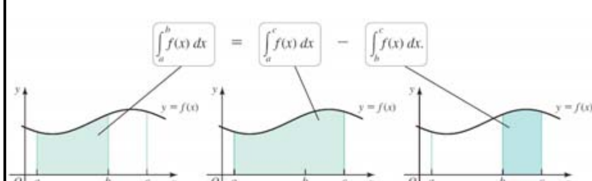


Figure 5.31 (1 of 2)

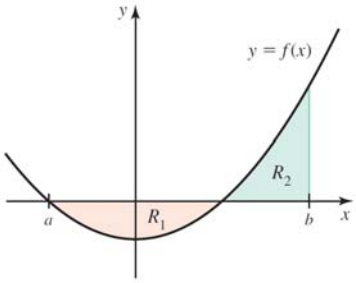
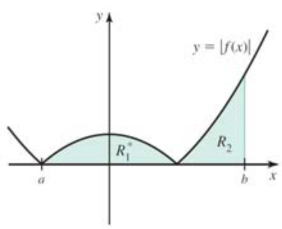


Figure 5.31 (2 of 2)



$$\int_a^b |f(x)|\,dx = \text{area of } R_1^* + \text{area of } R_2 = \text{area of } R_1 + \text{area of } R_2$$

Table 5.4

Let f and g be integrable functions on an interval that contains a, b, and c.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

5.3

Fundamental Theorem of Calculus

Figure 5.32

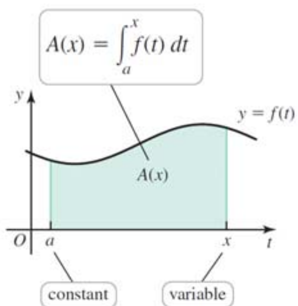
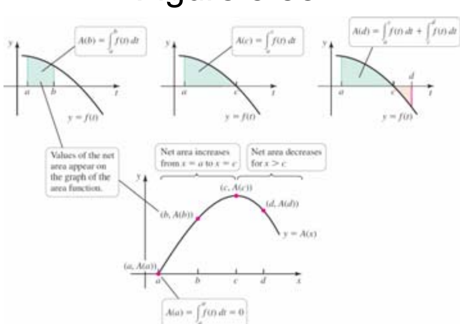


Figure 5.33



DEFINITION Area Function

Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Figure 5.34

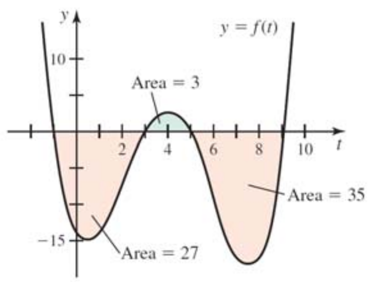


Figure 5.35

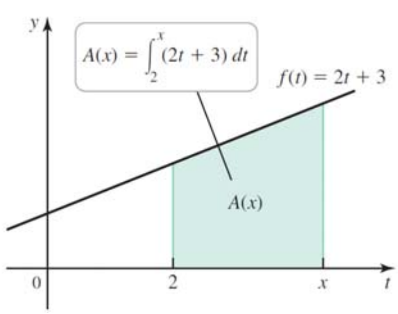


Figure 5.36

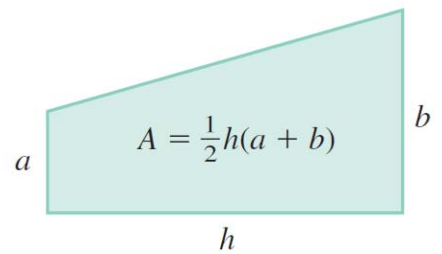


Figure 5.37

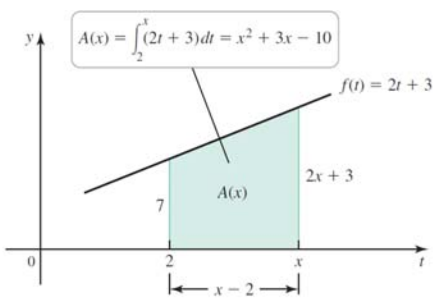


Figure 5.38

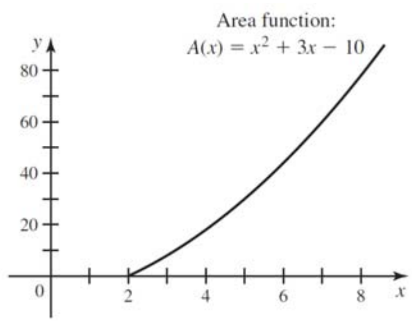
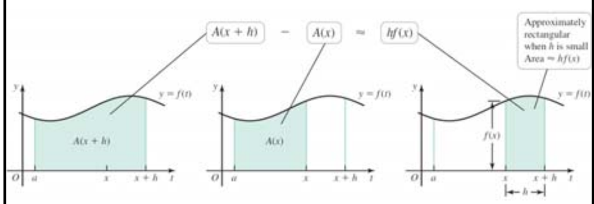


Figure 5.39



THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or, equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Figure 5.40

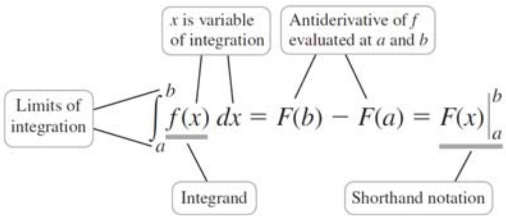


Figure 5.41

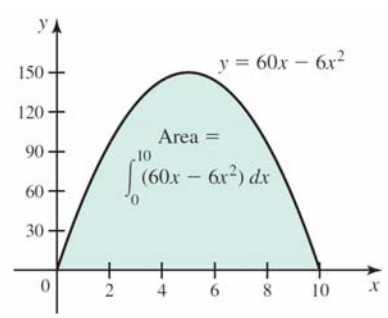


Figure 5.42

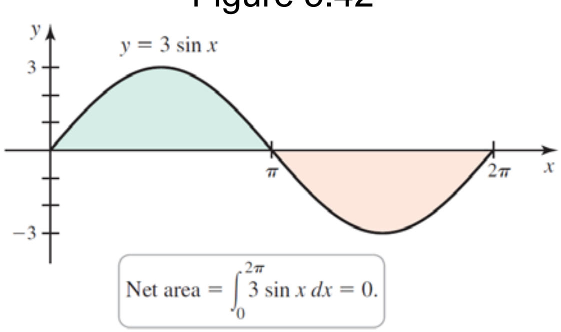


Figure 5.43

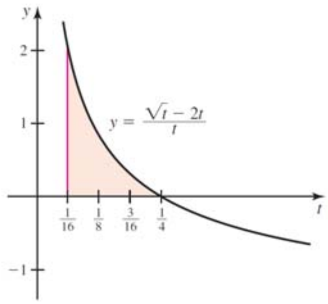


Figure 5.44

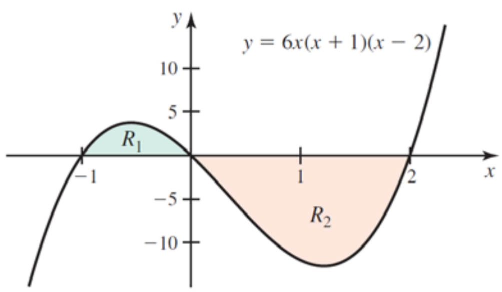


Figure 5.45

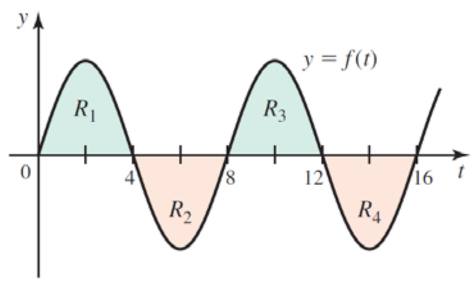


Figure 5.46 (a)

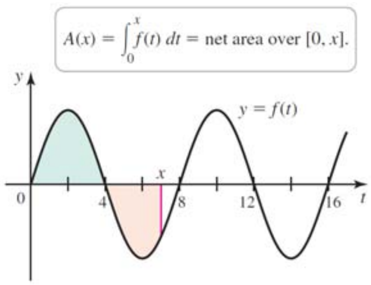


Figure 5.46 (b & c)

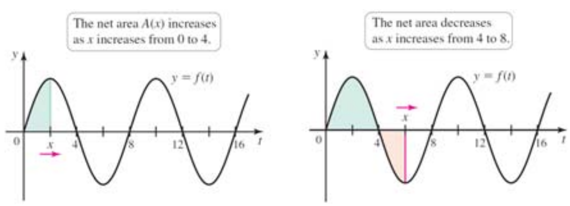


Figure 5.46 (d & e)

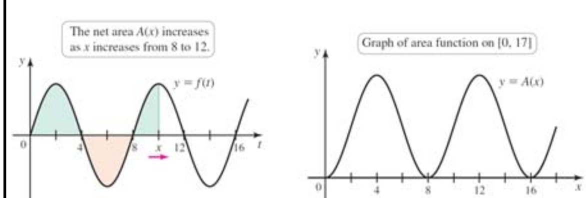


Figure 5.47

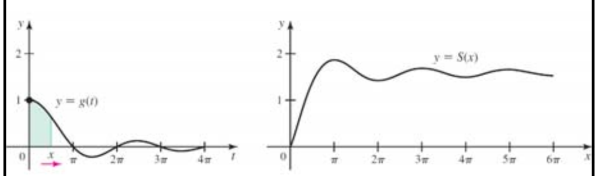


Figure 5.48

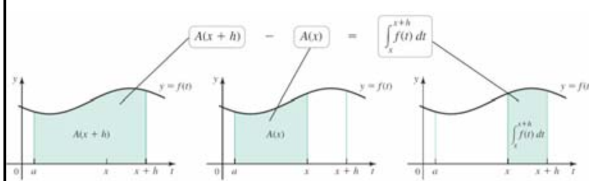


Figure 5.49 (1 of 2)

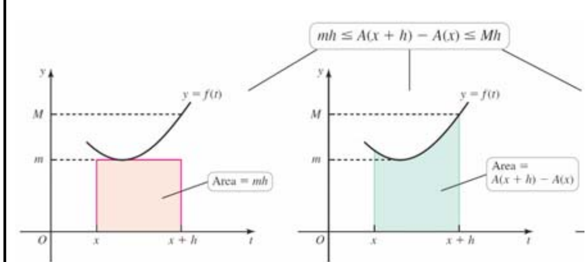
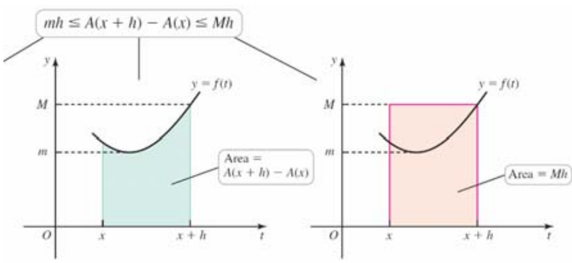


Figure 5.49 (2 of 2)



5.4

Working with Integrals

Figure 5.50 (a)

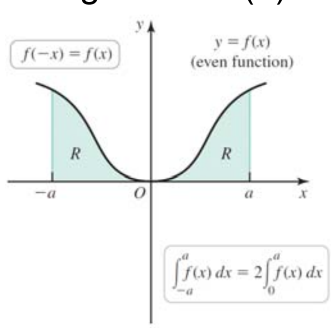
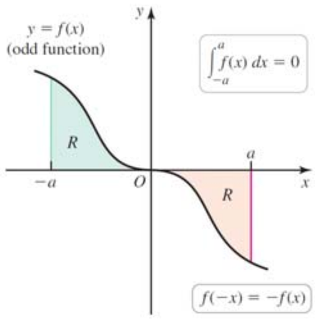


Figure 5.50 (b)



THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

DEFINITION Average Value of a Function

The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx.$$

Figure 5.51

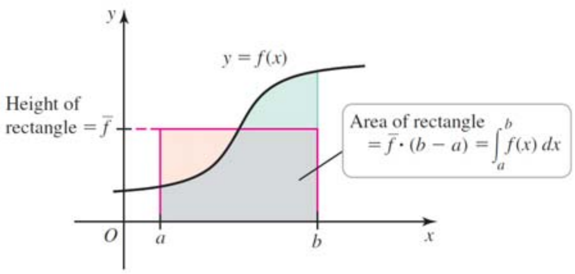


Figure 5.52

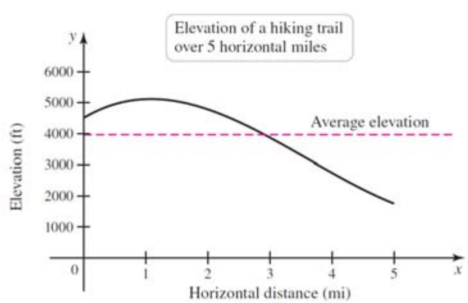
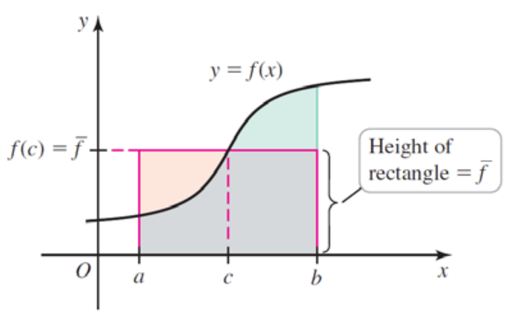


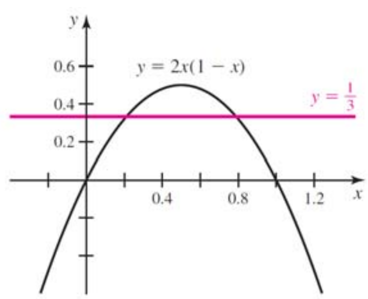
Figure 5.53



THEOREM 5.5 Mean Value Theorem for Integrals
Let f be continuous on the interval $[a, b]$. There exists a point c in $[a, b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Figure 5.54



5.5

Substitution Rule

THEOREM 5.6 Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

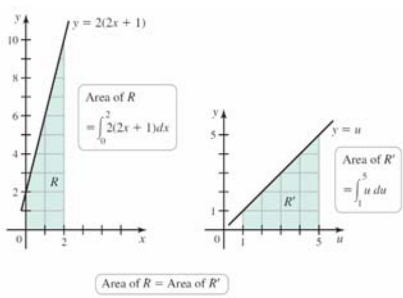
Disclaimer: Not all integrals yield to the Substitution Rule.

THEOREM 5.7 Substitution Rule for Definite Integrals

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Figure 5.55



y = 2(2x + 1)
Area of R
= ∫₀² 2(2x + 1)dx
R
y = u
Area of R′
= ∫₁⁵ u du
R′
Area of R = Area of R′