Chapter 5

Integration

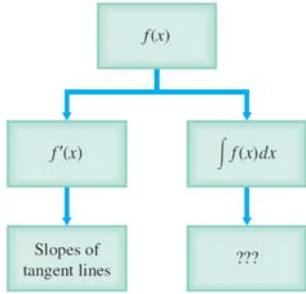Calculus
for Scientists and Engineers

BRIGGS   COCHRAN   GILLETT

5.1

Approximating Areas
under Curves

Figure 5.1

$f(x)$

$f'(x)$ $\int f(x)dx$

Slopes of
tangent lines   ???

1

Figure 5.2

Figure 5.3

Figure 5.4 (a)

Figure 5.4 (b)

$v = t^2$

$n = 4$

Figure 5.4 (c)

$v = t^2$

$n = 8$

Figure 5.5

$v = t^2$

$n = 64$

Table 5.1

2
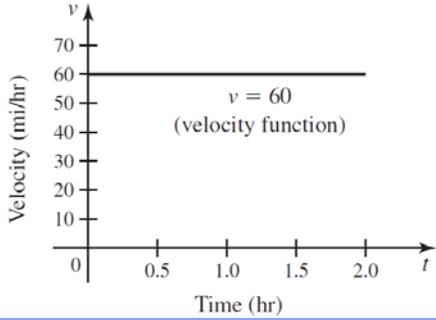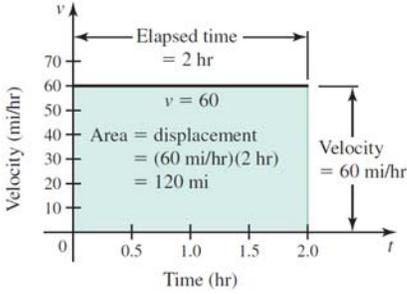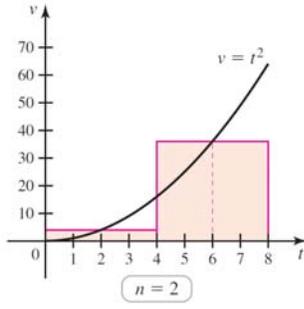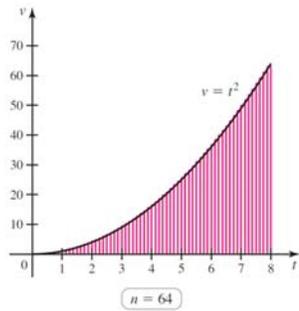
Approximations to the area under the velocity curve $v = t^2$ on $[0, 8]$

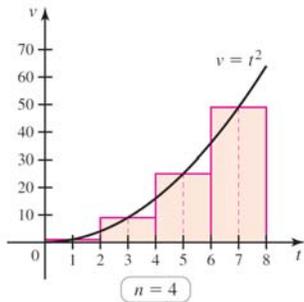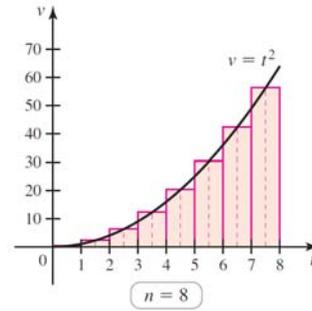| Number of subintervals	Length of each subinterval	Approximate displacement (area under curve)
1	8 s	128.0 m
2	4 s	160.0 m
4	2 s	168.0 m
8	1 s	170.0 m
16	0.5 s	170.5 m
32	0.25 s	170.625 m
64	0.125 s	170.65625 m

Figure 5.6

$y = f(x)$

R

O a b x

Figure 5.7

Δx Δx Δx $\cdots$ Δx

$x_0 = a$ x_1 x_2 x_3 $\cdots$ x_{n-1} $x_n = b$ x

DEFINITION Regular Partition

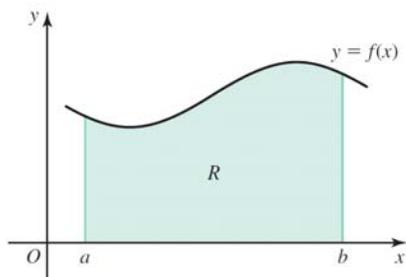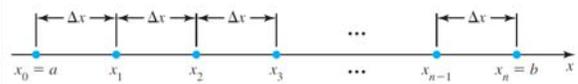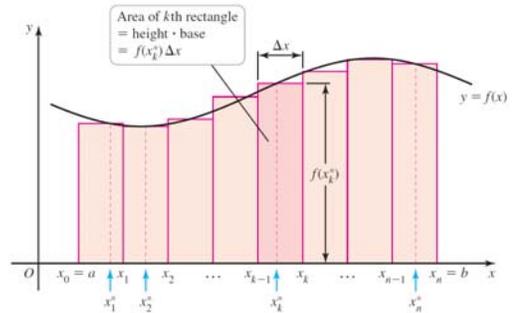Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \dfrac{b - a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \ldots,$ x_{n-1}, x_n of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \ldots, n.$$

Figure 5.8

DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is

• a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
• a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
• a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \ldots, n$.
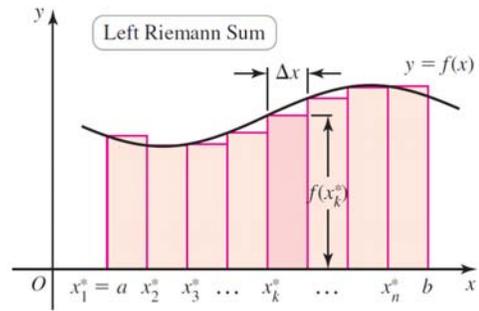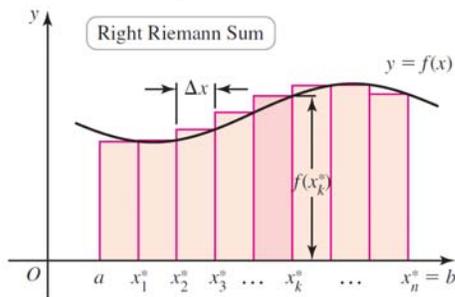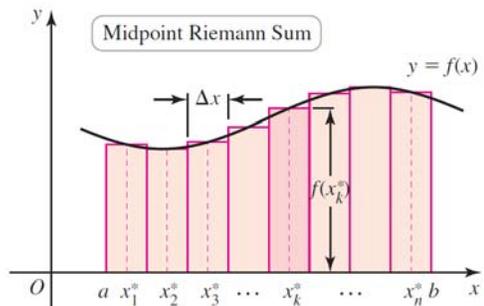
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Figure 5.9

Figure 5.10

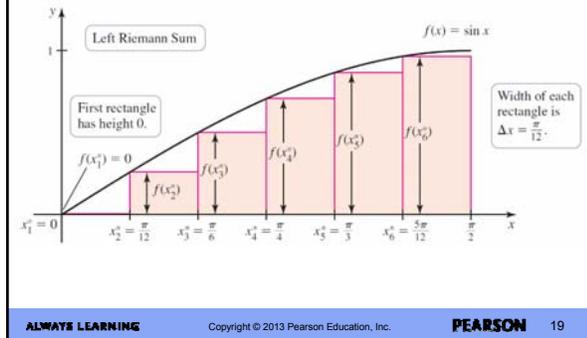## Figure 5.11

Figure 5.12

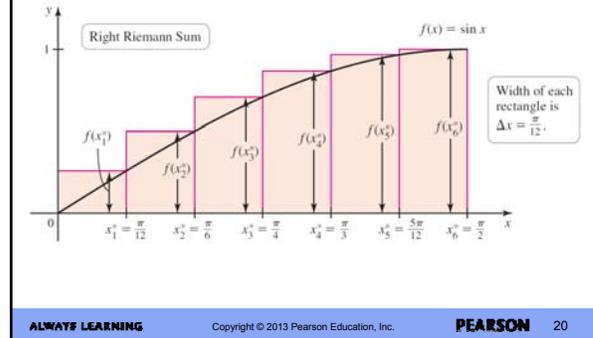## Figure 5.13
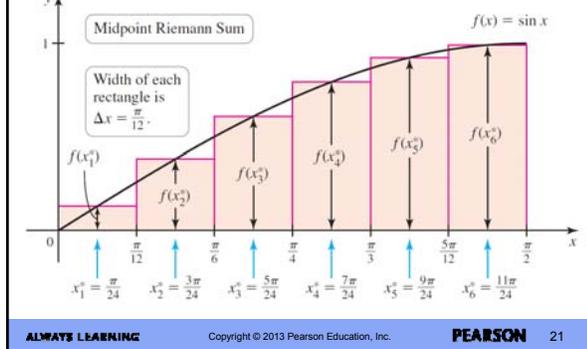
Figure 5.14

Table 5.2

x	$f(x)$
0	1
0.5	3
1.0	4.5
1.5	5.5
2.0	6.0

4

THEOREM 5.1   Sums of Positive Integers

Let n be a positive integer.

$$\sum_{k=1}^{n} c = cn \qquad\qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad\qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

• left Riemann sum if $x_k^* = a + (k-1)\Delta x$
• right Riemann sum if $x_k^* = a + k\Delta x$
• midpoint Riemann sum if $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, 2, \ldots, n$

Figure 5.15

$f(x) = x^3 + 1$

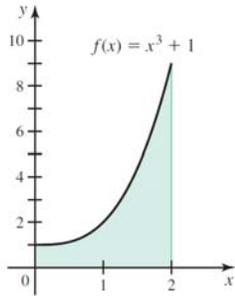## Table 5.3: Left, right, and midpoint Riemann sum approximations

n	L_n	R_n	M_n
20	5.61	6.41	5.995
40	5.8025	6.2025	5.99875
60	5.86778	6.13444	5.99944
80	5.90063	6.10063	5.99969
100	5.9204	6.0804	5.9998
120	5.93361	6.06694	5.99986
140	5.94306	6.05735	5.9999
160	5.95016	6.05016	5.99992
180	5.95568	6.04457	5.99994
200	5.9601	6.0401	5.99995

5.2

Definite Integrals

Figure 5.16

The midpoint Riemann sum for $f(x) = 1 - x^2$ on [1, 3] is −6.625.

$f(x) = 1 - x^2$

Figure 5.17

The left Riemann sum on [0, 1.5] is 0.875.

The left Riemann sum on [1.5, 3] is −4.75.

The resulting left Riemann sum on [0, 3] is −3.875.

$f(x) = 1 - x^2$

Figure 5.18

$y = f(x)$

The Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$ approximates the negative of the area of the region bounded between the x-axis and the curve.
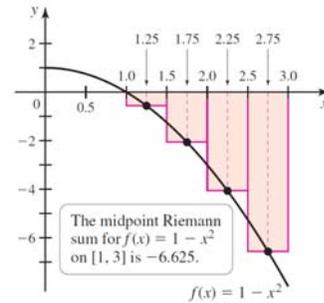
Figure 5.19

DEFINITION Net Area

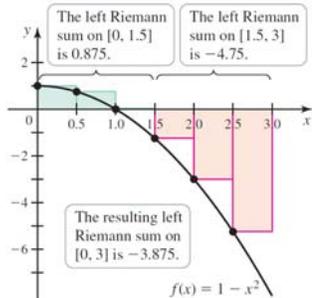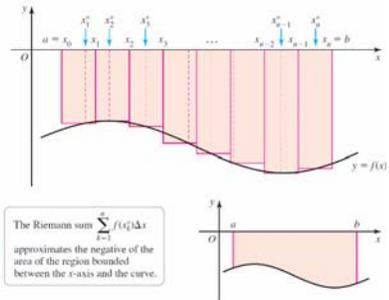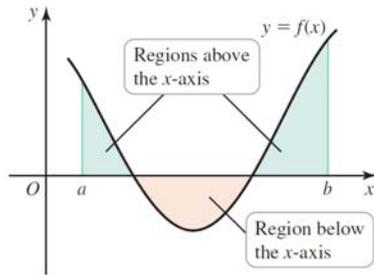Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.
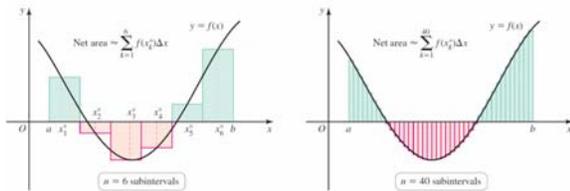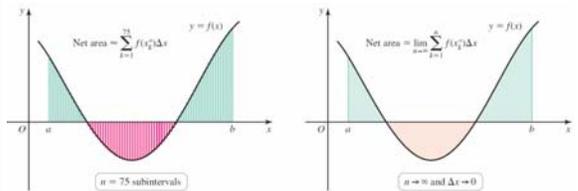

Figure 5.20 (1 of 2)
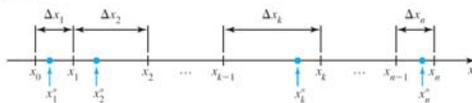

Figure 5.20 (2 of 2)

DEFINITION General Riemann Sum

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Figure 5.21

Upper limit of integration

Upper limit of summation

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Integrand

Lower limit of integration

Lower limit of summation

x is the variable of integration.

Figure 5.22

$$\int_a^b f(x)\,dx \;=\; \int_a^b f(\omega)\,d\omega \;=\; \int_a^b f(s)\,ds$$

$y = f(x)$

$y = f(\omega)$

$y = f(s)$
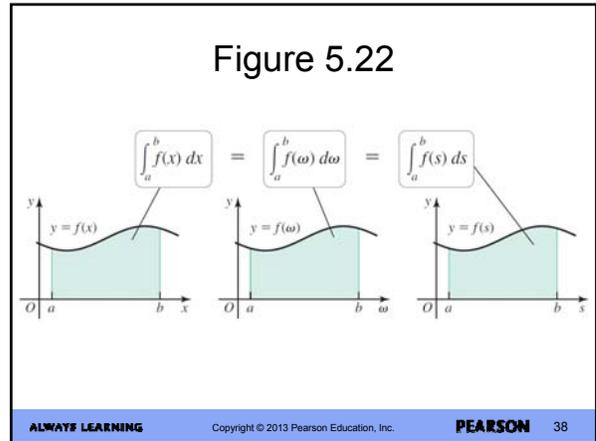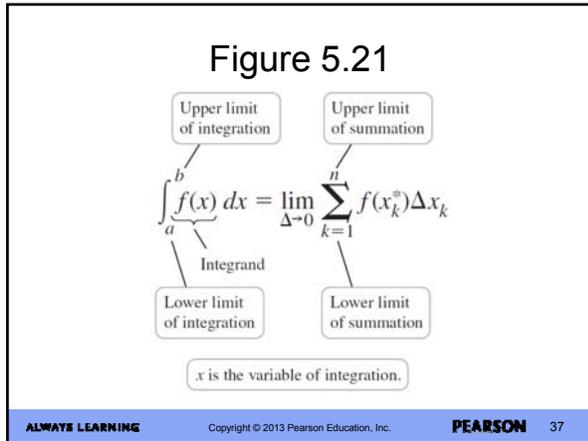
THEOREM 5.2   Integrable Functions
If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

7

Figure 5.23

Net area $= \int_a^b f(x)\,dx$

= area above x-axis (Regions 1 and 3)
− area below x-axis (Region 2)

A bounded piecewise continuous function is integrable.

Figure 5.24

$y = 3x^2 + 2x + 1$

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (3x_k^{*2} + 2x_k^* + 1)\Delta x_k = \int_1^3 (3x^2 + 2x + 1)\,dx$$

Figure 5.25

$y = 2x + 3$

Area = 18

11

7

2

Figure 5.26

$y = 2x - 6$

Area = 9

Area = 4

Figure 5.27

$y = \sqrt{1 - (x-3)^2}$

Area of shaded region $= \frac{1}{4}\pi(1)^2 = \frac{1}{4}\pi$

Figure 5.28

$y = f(x)$

12
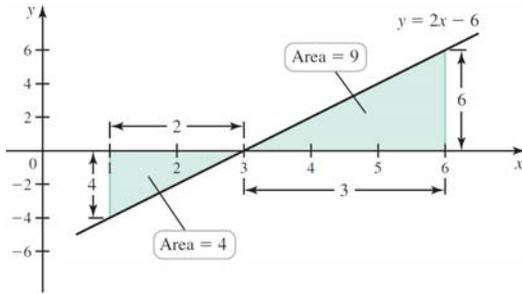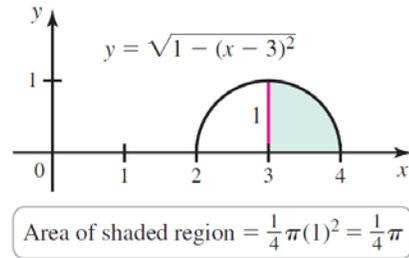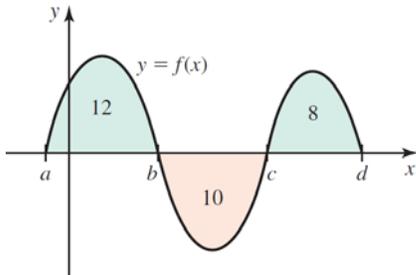
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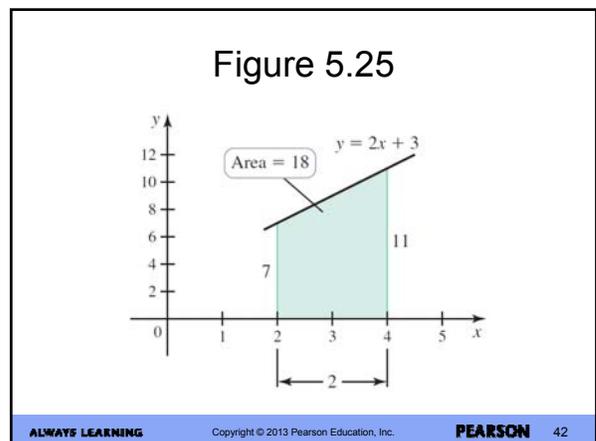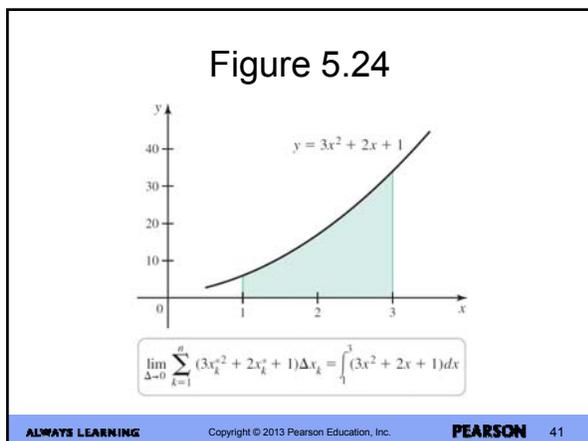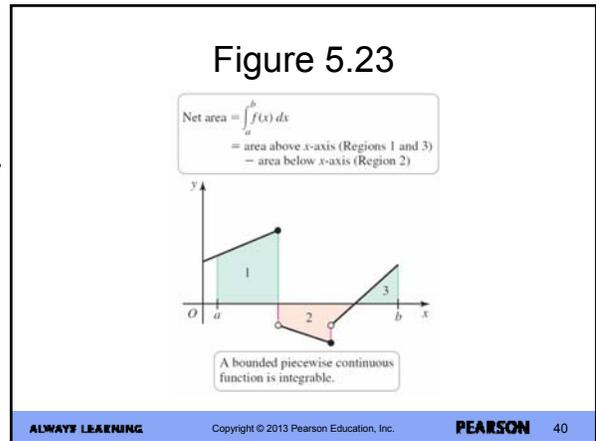
8

DEFINITION Reversing Limits and Identical Limits

Suppose f is integrable on $[a, b]$.

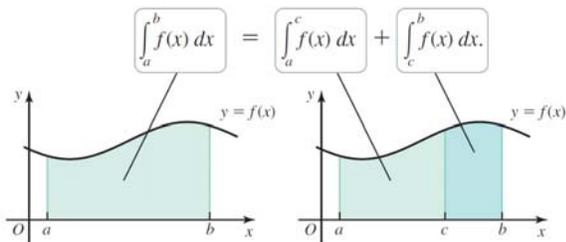1. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$　　　2. $\int_a^a f(x)\, dx = 0$

Figure 5.29

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$y = f(x)$

$y = f(x)$

Figure 5.30

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx - \int_b^c f(x)\, dx.$$
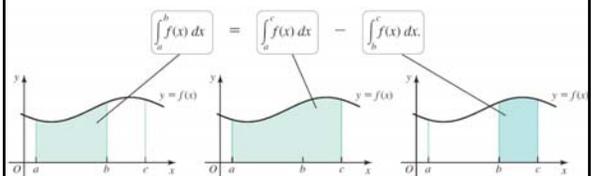
$y = f(x)$

$y = f(x)$

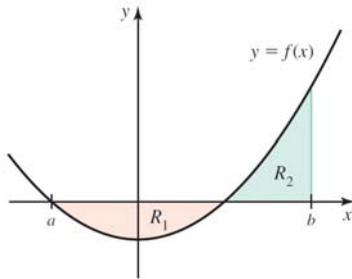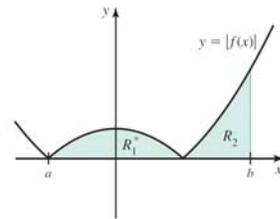$y = f(x)$

Figure 5.31 (1 of 2)

Figure 5.31 (2 of 2)

$$\int_a^b |f(x)|\, dx = \text{area of } R_1^* + \text{area of } R_2$$
$$= \text{area of } R_1 + \text{area of } R_2$$

Table 5.4

Let f and g be integrable functions on an interval that contains a, b, and c.

1. $\int_a^a f(x)\, dx = 0$ Definition

2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ Definition

3. $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

4. $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ For any constant c

5. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\, dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

9

5.3

Fundamental Theorem of Calculus

Figure 5.32

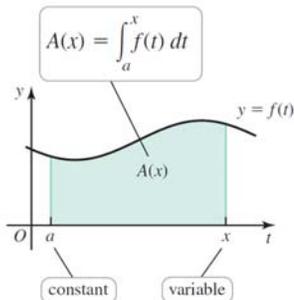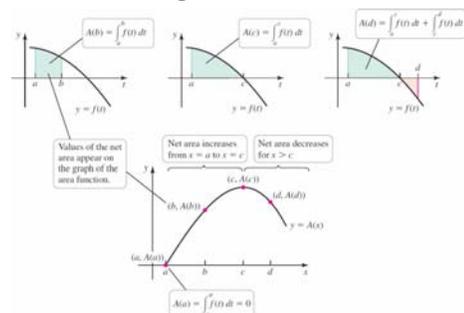$$A(x) = \int_a^x f(t)\, dt$$

Figure 5.33

DEFINITION Area Function

Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint** a is

$$A(x) = \int_a^x f(t)\, dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.
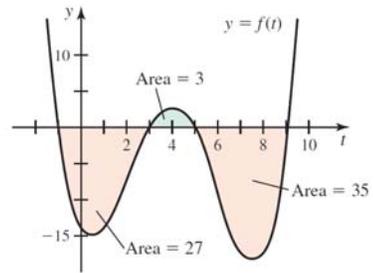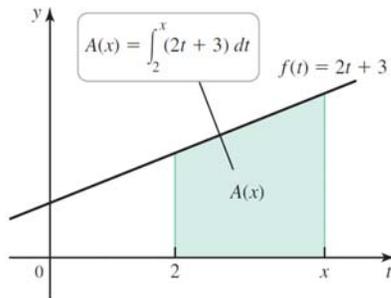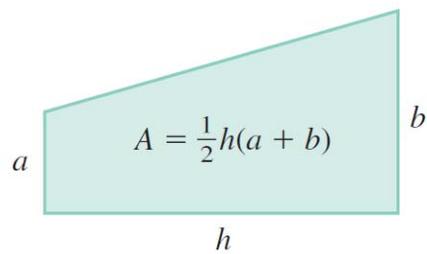

Figure 5.34

Area = 3

Area = 35

Area = 27

$y = f(t)$

Figure 5.35

$$A(x) = \int_2^x (2t + 3)\, dt$$

$f(t) = 2t + 3$

$A(x)$

Figure 5.36

$$A = \frac{1}{2}h(a + b)$$

a b

h

Figure 5.37

$$A(x) = \int_2^x (2t + 3)\, dt = x^2 + 3x - 10$$

$f(t) = 2t + 3$

$2x + 3$

$A(x)$

7

$x - 2$

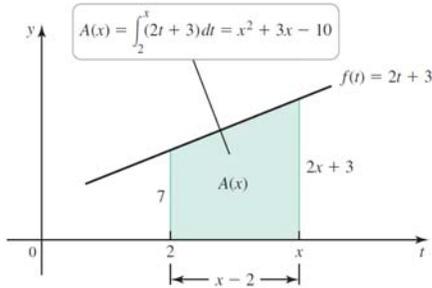## Figure 5.38

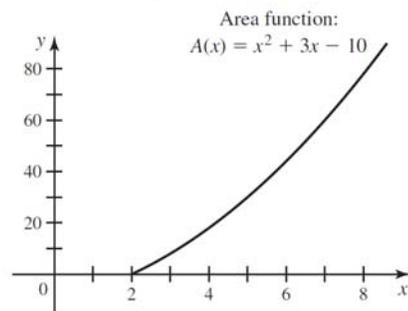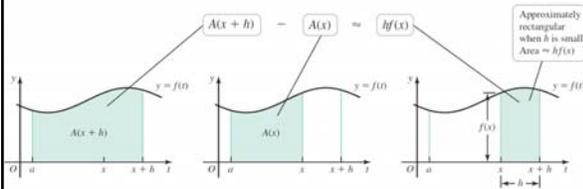Area function:

$$A(x) = x^2 + 3x - 10$$

Figure 5.39

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\, dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or, equivalently,
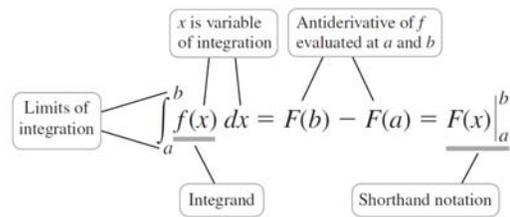
$$A'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Figure 5.40

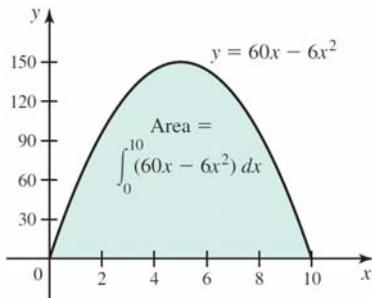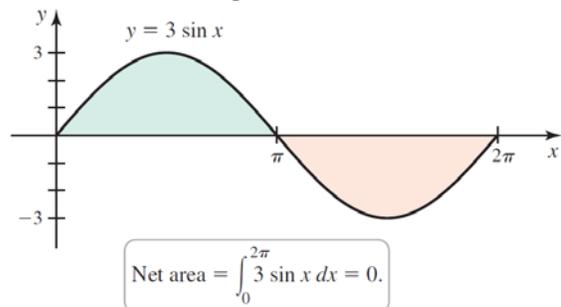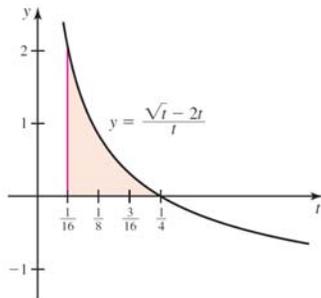---

Figure 5.41

Figure 5.42

Figure 5.43
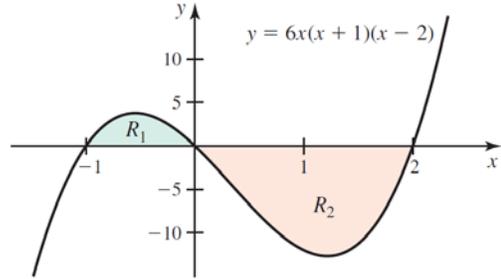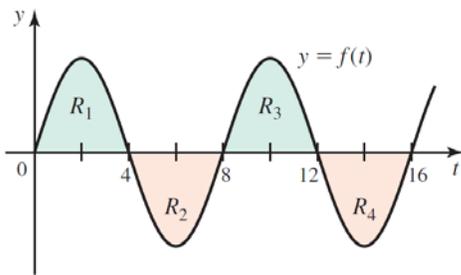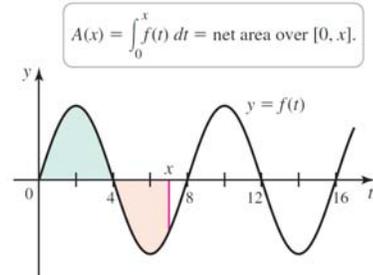
$$y = \frac{\sqrt{t} - 2t}{t}$$

Figure 5.44

$$y = 6x(x + 1)(x - 2)$$

R_1

R_2

Figure 5.45

$y = f(t)$

R_1　R_3

R_2　R_4

Figure 5.46 (a)

$$A(x) = \int_0^x f(t)\, dt = \text{net area over } [0, x].$$

$y = f(t)$

Figure 5.46 (b & c)

The net area $A(x)$ increases as x increases from 0 to 4.

The net area decreases as x increases from 4 to 8.

$y = f(t)$

Figure 5.46 (d & e)

The net area $A(x)$ increases as x increases from 8 to 12.

Graph of area function on [0, 17]

$y = f(t)$

$y = A(x)$

Figure 5.47

Figure 5.48

$$A(x+h) \quad - \quad A(x) \quad = \quad \int_x^{x+h} f(t)\, dt$$

Figure 5.49 (1 of 2)

$$mh \le A(x+h) - A(x) \le Mh$$

Area = mh

$$\text{Area} = A(x+h) - A(x)$$

Figure 5.49 (2 of 2)

$$mh \le A(x+h) - A(x) \le Mh$$

$$\text{Area} = A(x+h) - A(x)$$

Area = Mh
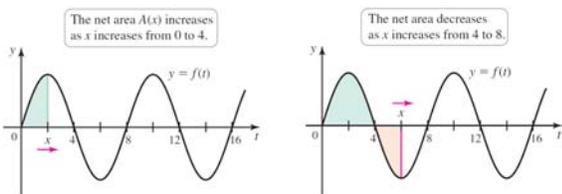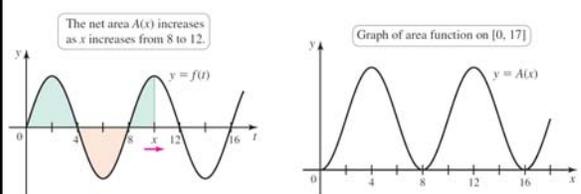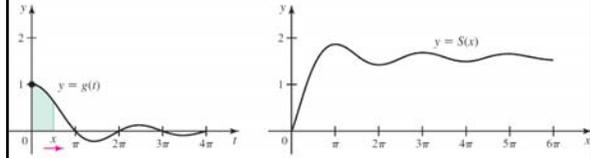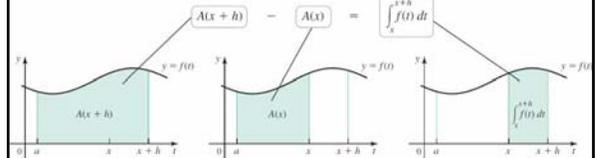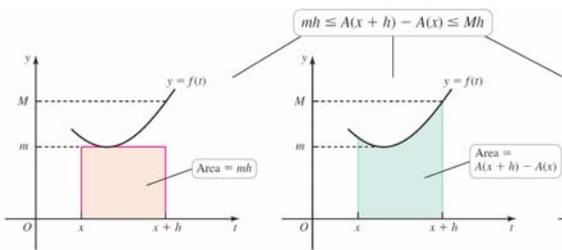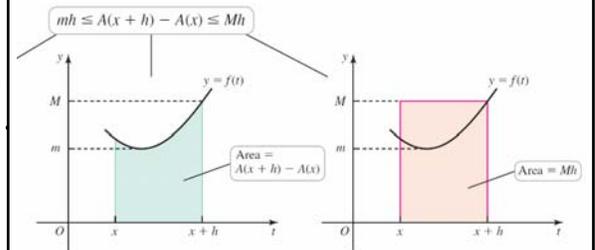
5.4

Working with Integrals

Figure 5.50 (a)

$$f(-x) = f(x)$$
$$y = f(x)$$
(even function)

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$$

Figure 5.50 (b)

$y = f(x)$
(odd function)

$\int_{-a}^{a} f(x)\, dx = 0$

R

$-a$ O a x

R

$f(-x) = -f(x)$

THEOREM 5.4   Integrals of Even and Odd Functions
Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.
- If f is even, $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.
- If f is odd, $\int_{-a}^{a} f(x)\, dx = 0$.

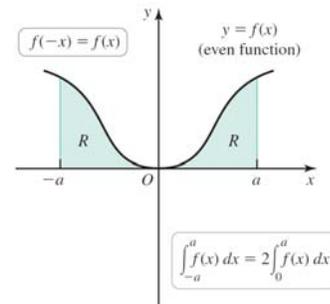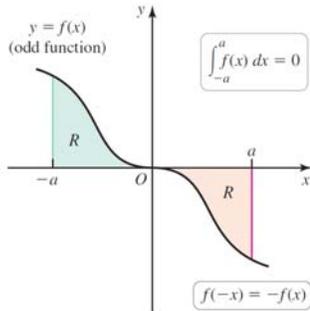**DEFINITION Average Value of a Function**
The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\, dx.$$

Figure 5.51

$y = f(x)$

Height of rectangle $= \bar{f}$

Area of rectangle $= \bar{f} \cdot (b - a) = \int_{a}^{b} f(x)\, dx$

O a b x

Figure 5.52

Elevation of a hiking trail over 5 horizontal miles

Average elevation

Elevation (ft)

Horizontal distance (mi)

Figure 5.53

$y = f(x)$

$f(c) = \bar{f}$

Height of rectangle $= \bar{f}$

O a c b x
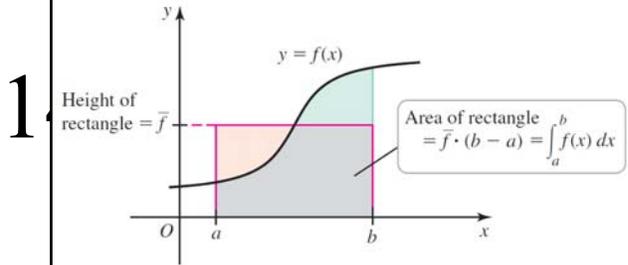
THEOREM 5.5 Mean Value Theorem for Integrals

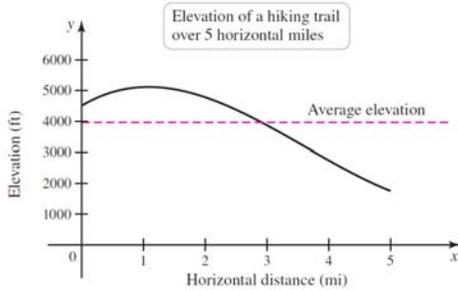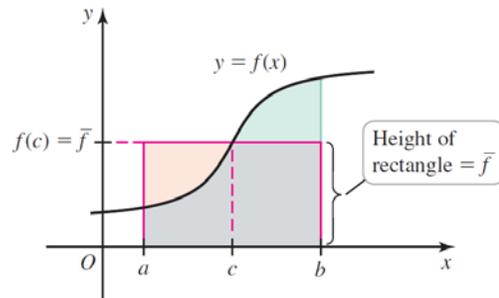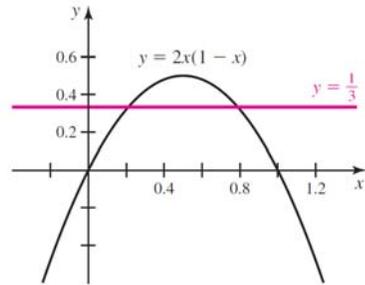Let f be continuous on the interval $[a, b]$. There exists a point c in $[a, b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Figure 5.54

5.5

Substitution Rule

1

THEOREM 5.6 Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.

2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.

3. Evaluate the new indefinite integral with respect to u.

4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

THEOREM 5.7 Substitution Rule for Definite Integrals

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Figure 5.55

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