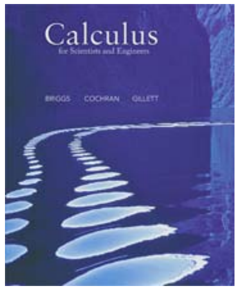
Chapter 6

Applications of Integration



6.1

Velocity and Net Change

1

Figure 6.1

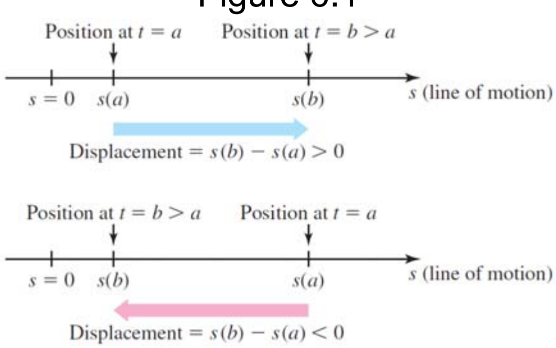
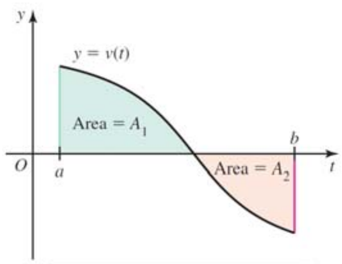
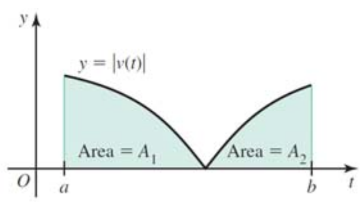


Figure 6.2 (a)



Displacement $= A_1 - A_2 = \int_a^b v(t)\,dt$

Figure 6.2 (b)

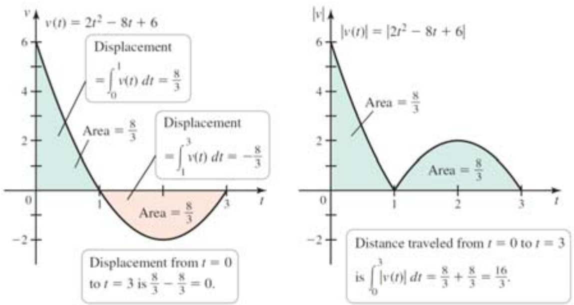


Distance traveled $= A_1 + A_2 = \int_a^b |v(t)|\,dt$

DEFINITION Position, Velocity, Displacement, and Distance

1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is
$$s(b) - s(a) = \int_a^b v(t)\,dt.$$
4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is
$$\int_a^b |v(t)|\,dt,$$
where $|v(t)|$ is the **speed** of the object at time t.

Figure 6.3



THEOREM 6.1 Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position at}\\ \text{time } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,\,t]}}.$$

3

Figure 6.4

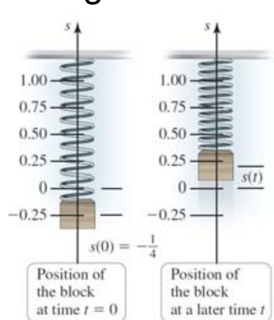


Figure 6.5 (a)

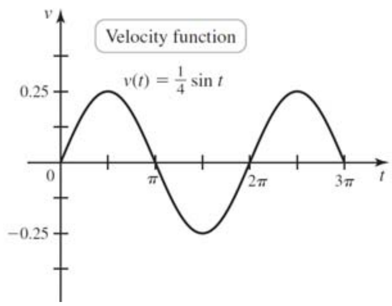


Figure 6.5 (b)

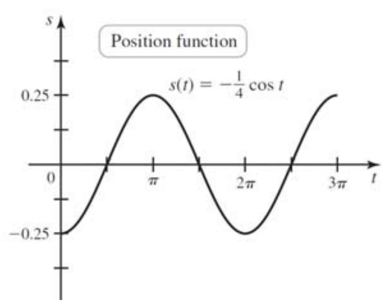
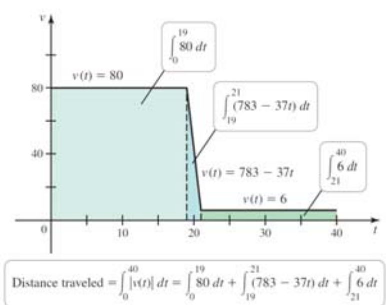


Figure 6.6



THEOREM 6.2 Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx.$$

Figure 6.7

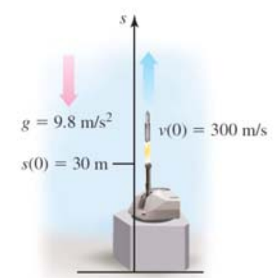
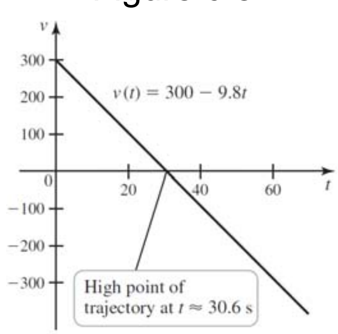


Figure 6.8



THEOREM 6.3 Net Change and Future Value

Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\,dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx.$$

Table 6.1

Velocity–Displacement Problems	General Problems
Position $s(t)$	Quantity $Q(t)$ (such as volume or population size)
Velocity: $s'(t) = v(t)$	Rate of change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$

Figure 6.9

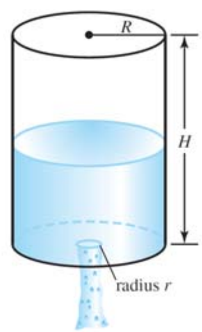
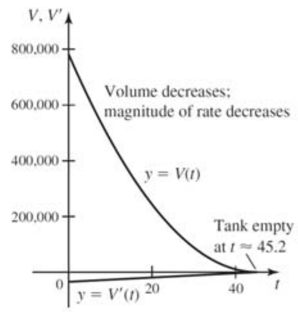


Figure 6.10



6.2

Regions Between Curves

Figure 6.11

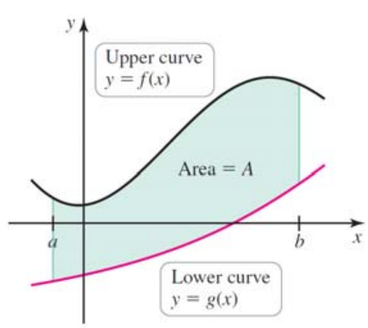


Figure 6.12

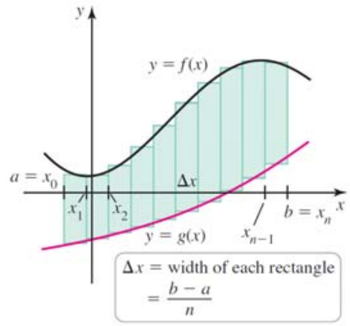
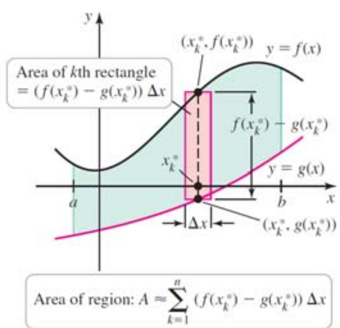


Figure 6.13



DEFINITION Area of a Region Between Two Curves

Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))\,dx.$$

Figure 6.14

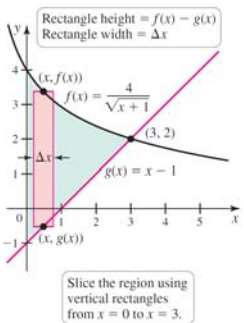


Figure 6.15 (a)

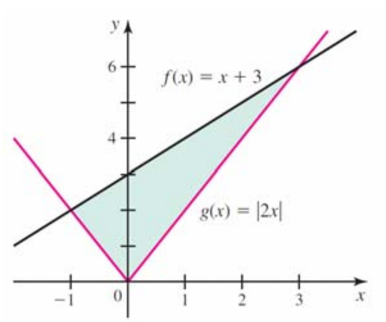


Figure 6.15 (b)

$$g(x) = |2x| = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$$

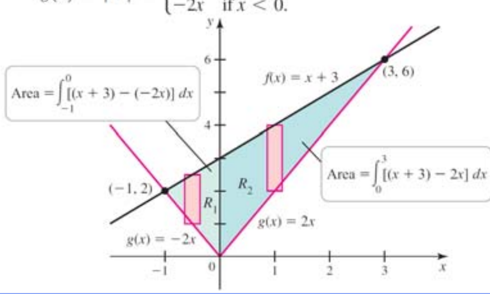


Figure 6.16

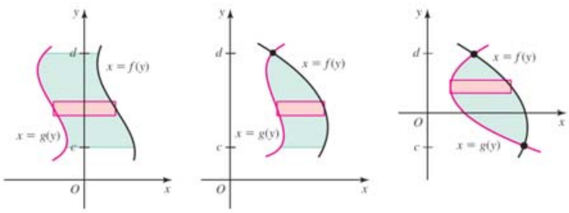
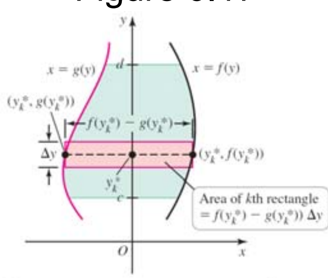


Figure 6.17



Area of region: $A \approx \sum_{k=1}^{n} (f(y_k^*) - g(y_k^*))\,\Delta y$

DEFINITION Area of a Region Between Two Curves with Respect to y

Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\,dy.$$

Figure 6.18

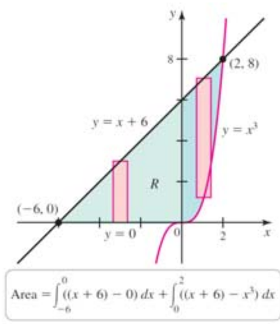


Figure 6.19

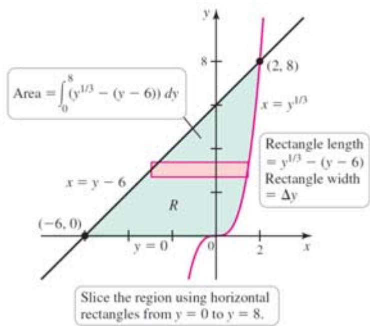
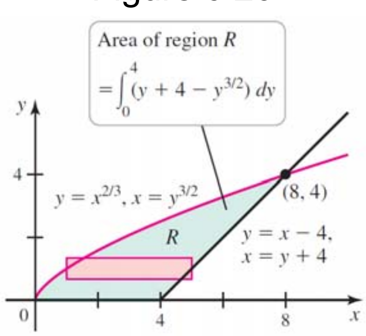


Figure 6.20



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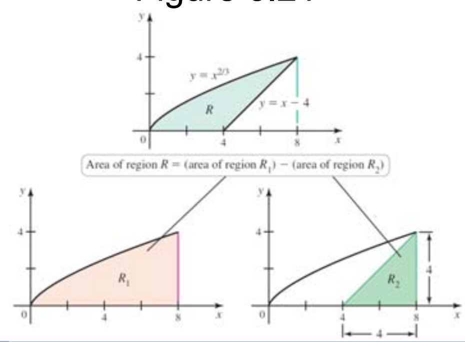


6.3

Volume by Slicing

Figure 6.22

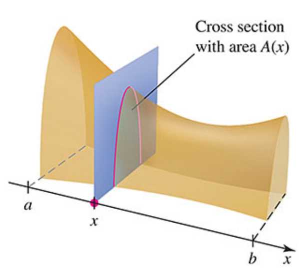


Figure 6.23

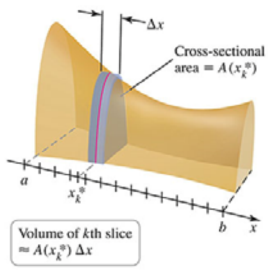
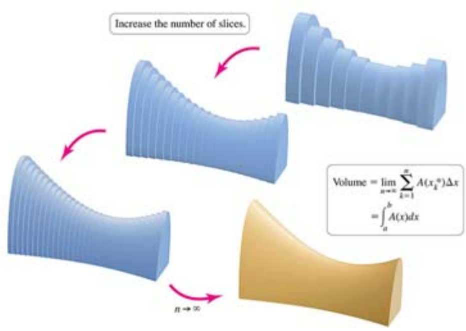


Figure 6.24



General Slicing Method

Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx.$$

Figure 6.25

$$r = \frac{1}{2}((2 - x^2) - x^2) = 1 - x^2.$$

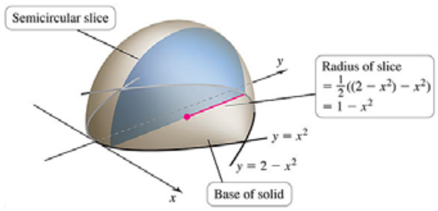


Figure 6.26

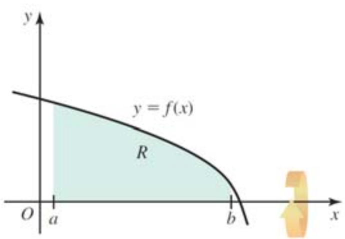


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Figure 6.27

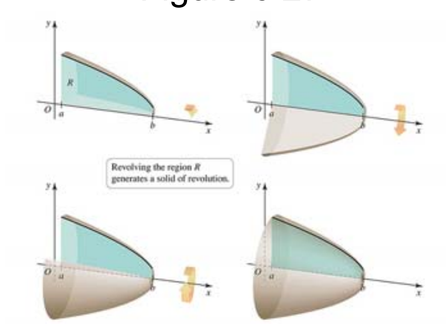
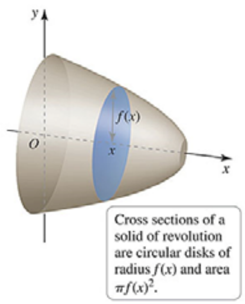


Figure 6.28



Disk Method About the x-Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2 \, dx.$$

Figure 6.29 (1 of 2)

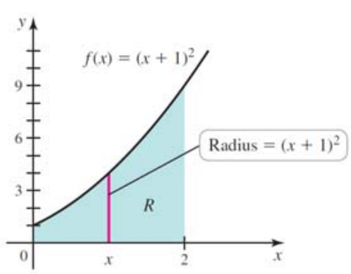


Figure 6.29 (2 of 2)

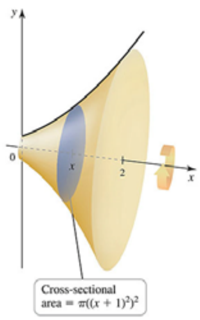


Figure 6.30

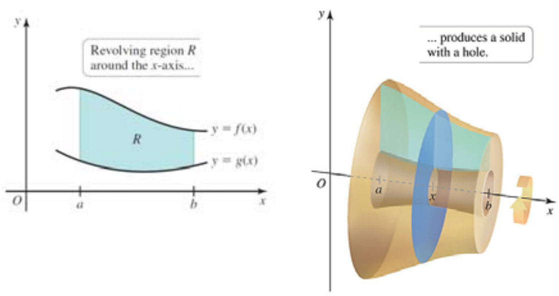


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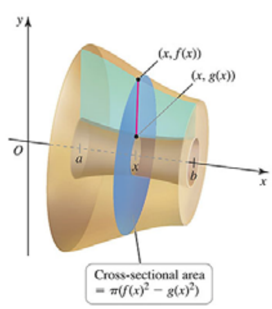
Figure 6.31



Washer Method About the x-Axis

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)\, dx.$$

Figure 6.32

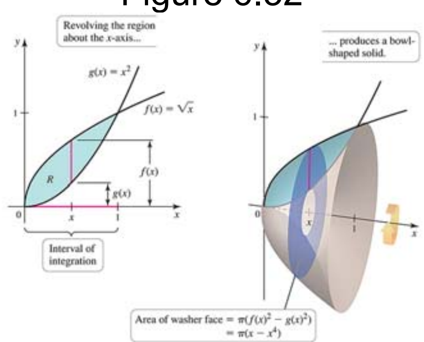
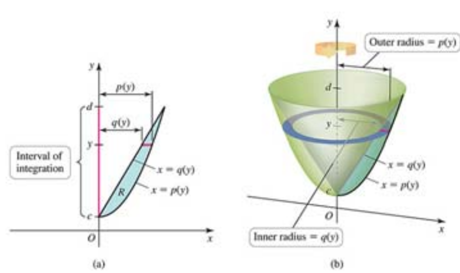


Figure 6.33 (a & b)



Disk and Washer Methods About the *y*-Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi(p(y)^2 - q(y)^2)\,dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\,dy.$$

Figure 6.34

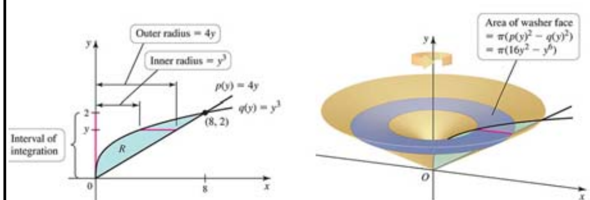
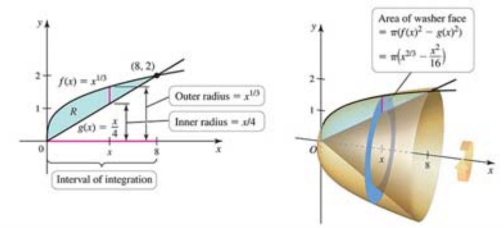


Figure 6.35



6.4

Volume by Shells

Figure 6.36

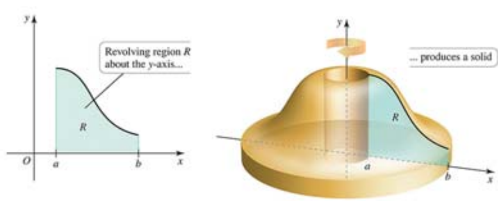


Figure 6.37

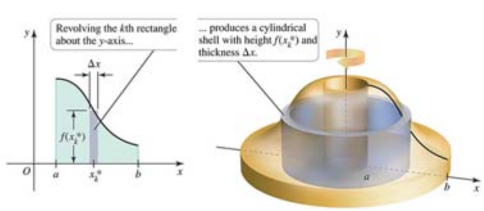


Figure 6.38

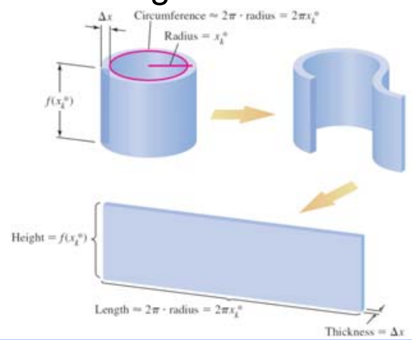


Figure 6.39

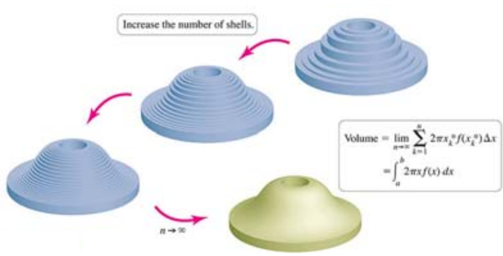
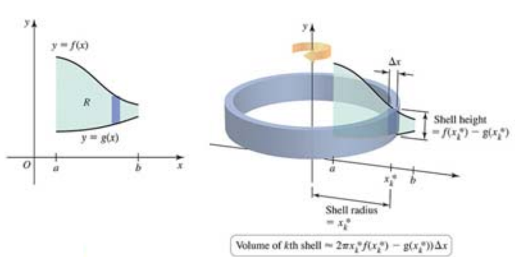


Figure 6.40



Volume by the Shell Method

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x(f(x) - g(x))\, dx.$$

Figure 6.41

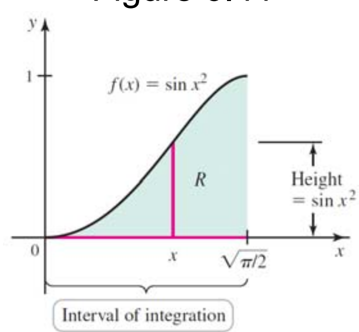


Figure 6.42

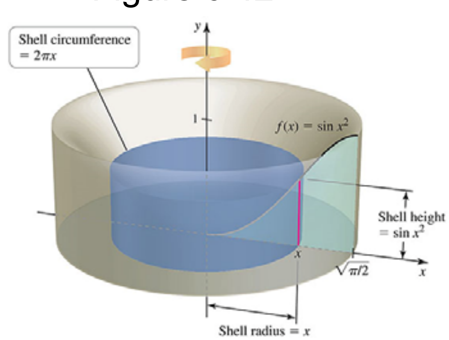


Figure 6.43

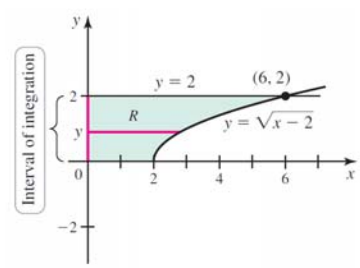


Figure 6.44

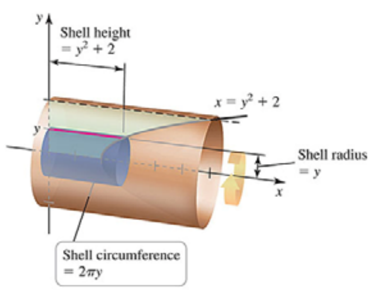


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Figure 6.45 (a & b)

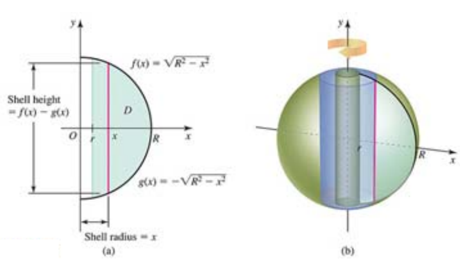


Figure 6.46 (a)

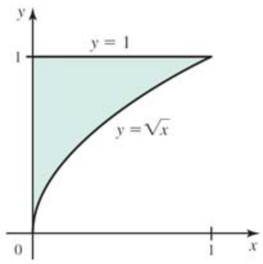


Figure 6.46 (b)

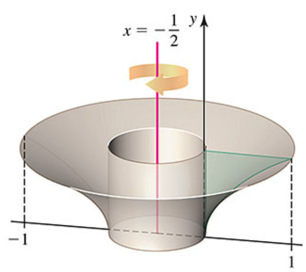


Figure 6.46 (c)

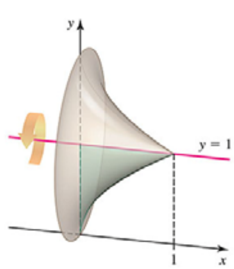


Figure 6.47

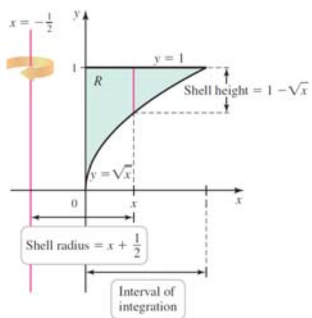
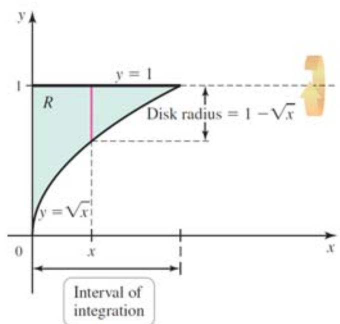
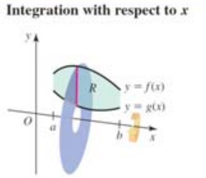
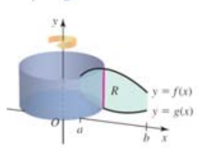


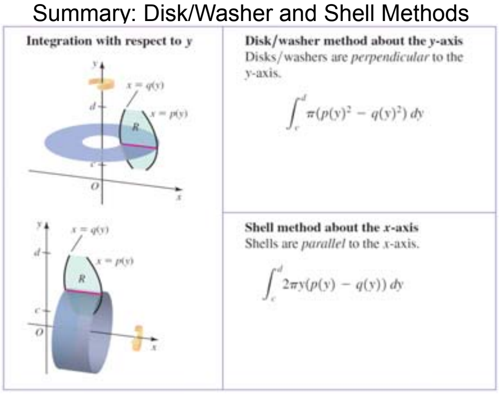
Figure 6.48



Summary: Disk/Washer and Shell Methods *continued…*

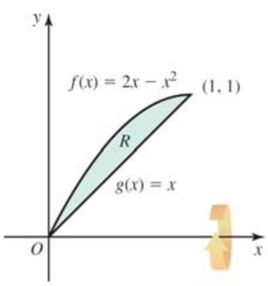
Integration with respect to x	
	**Disk / washer method about the x-axis** Disks / washers are *perpendicular* to the x-axis. $\int_a^b \pi(f(x)^2 - g(x)^2)\,dx$
	**Shell method about the y-axis** Shells are *parallel* to the y-axis. $\int_a^b 2\pi x(f(x) - g(x))\,dx$

Summary: Disk/Washer and Shell Methods



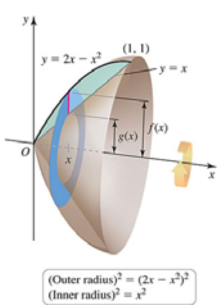
 73

Figure 6.49



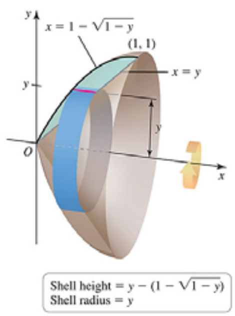
 74

Figure 6.50



 75

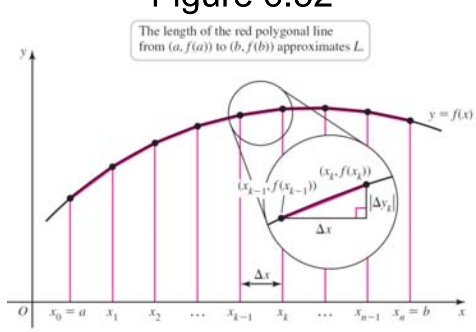
Figure 6.51



6.5

Length of Curves

Figure 6.52



DEFINITION Arc Length for $y = f(x)$

Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Figure 6.53

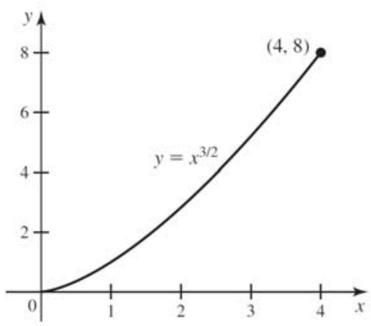
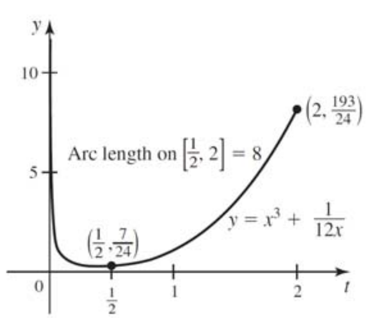


Figure 6.54

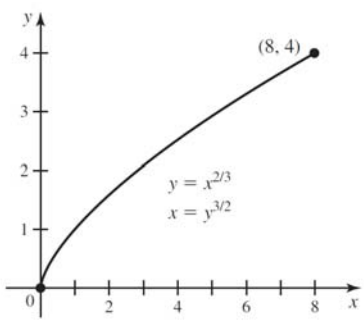


DEFINITION Arc Length for $x = g(y)$

Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Figure 6.55



6.6

Surface Area

Figure 6.56

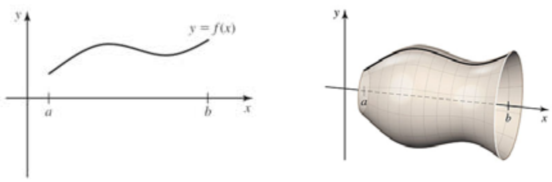


Figure 6.57

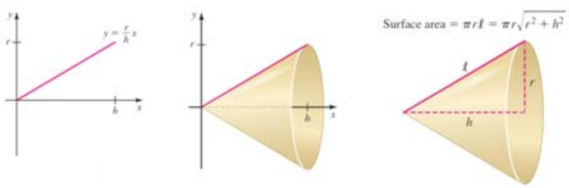


Figure 6.58

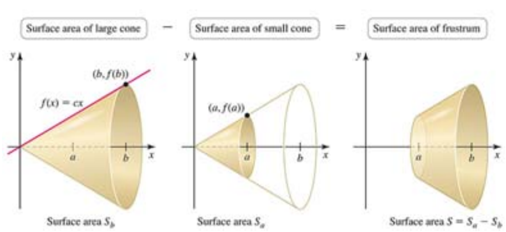


Figure 6.59

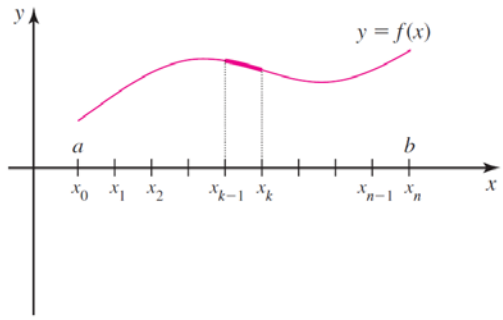
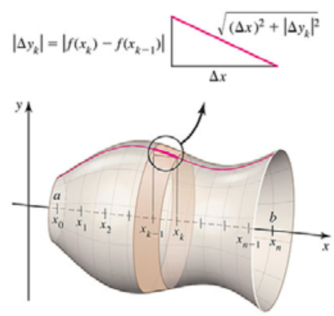


Figure 6.60



DEFINITION Area of a Surface of Revolution

Let f be differentiable and positive on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx.$$

Figure 6.61

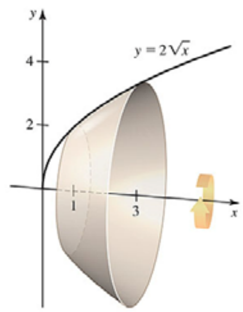


Figure 6.62

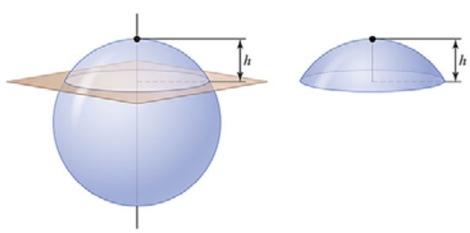


Figure 6.63

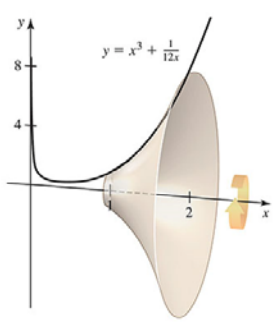
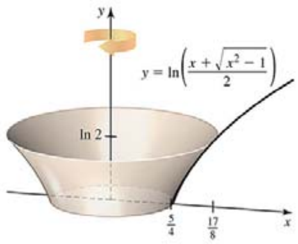


Figure 6.64



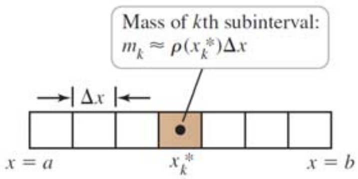
6.7

Physical Applications

32

Figure 6.65

Figure 6.66

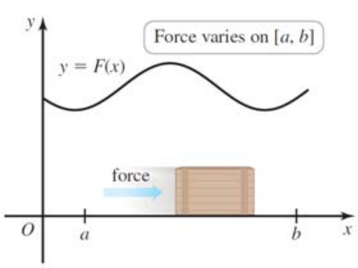


DEFINITION Mass of a One-Dimensional Object

Suppose a thin bar or wire is represented by a line segment on the interval $a \leq x \leq b$ with a density function ρ (with units of mass per length). The **mass** of the object is

$$m = \int_a^b \rho(x)\,dx.$$

Figure 6.67



DEFINITION Work

The work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx.$$

Figure 6.68

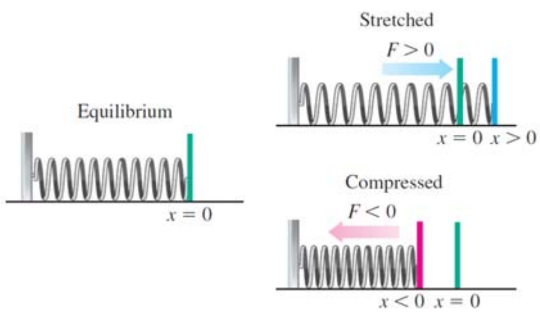


Figure 6.69

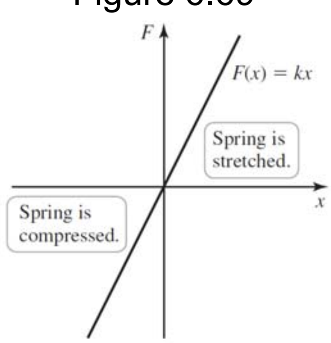
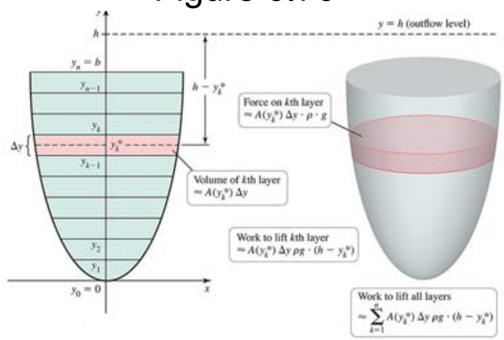


Figure 6.70



PROCEDURE Solving Lifting Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.
2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

$$W = \int_a^b \rho g A(y) D(y)\, dy.$$

Figure 6.71

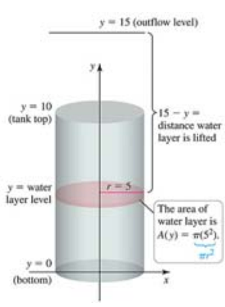


Figure 6.72

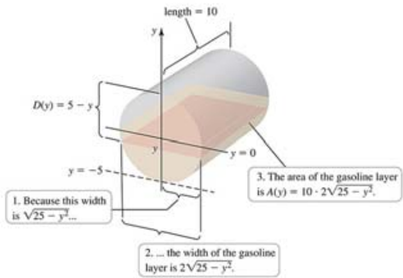
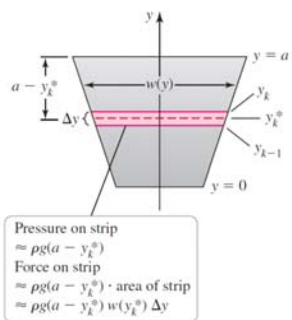


Figure 6.73



PROCEDURE Solving Force/Pressure Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam.
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a - y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}}\, dy.$$

Figure 6.74

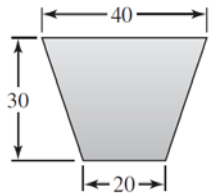


Figure 6.75