Chapter 6

Applications of Integration

Calculus
for Scientists and Engineers
BRIGGS COCHRAN GILLETT
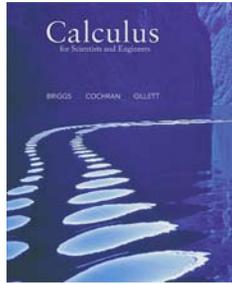
6.1

Velocity and Net Change

Figure 6.1

Figure 6.2 (a)

$$\text{Displacement} = A_1 - A_2 = \int_a^b v(t)\, dt$$

Figure 6.2 (b)

$$\text{Distance traveled} = A_1 + A_2 = \int_a^b |v(t)|\, dt$$

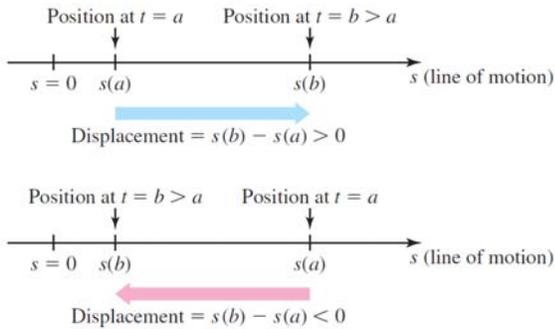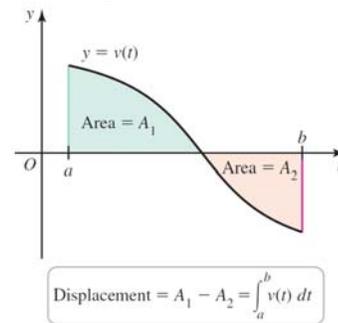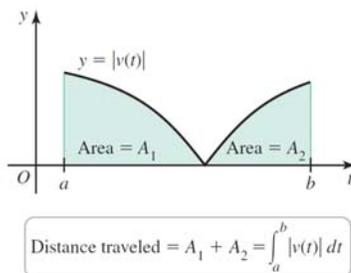**DEFINITION Position, Velocity, Displacement, and Distance**

1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
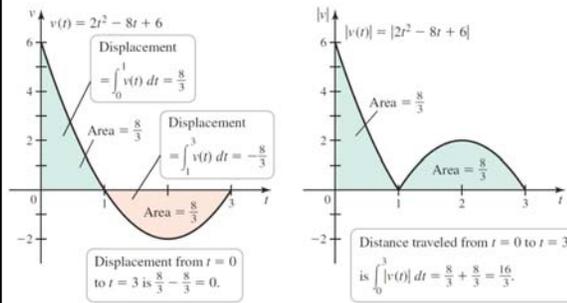3. The **displacement** of the object between $t = a$ and $t = b > a$ is

$$s(b) - s(a) = \int_a^b v(t)\, dt.$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)|\, dt,$$

where $|v(t)|$ is the **speed** of the object at time t.

Figure 6.3

$v(t) = 2t^2 - 8t + 6$

Displacement $= \int_0^1 v(t)\, dt = \frac{8}{3}$

Area $= \frac{8}{3}$

Displacement $= \int_1^3 v(t)\, dt = -\frac{8}{3}$

Area $= \frac{8}{3}$

Displacement from $t = 0$ to $t = 3$ is $\frac{8}{3} - \frac{8}{3} = 0$.

$|v(t)| = |2t^2 - 8t + 6|$

Area $= \frac{8}{3}$

Area $= \frac{8}{3}$

Distance traveled from $t = 0$ to $t = 3$ is $\int_0^3 |v(t)|\, dt = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$.
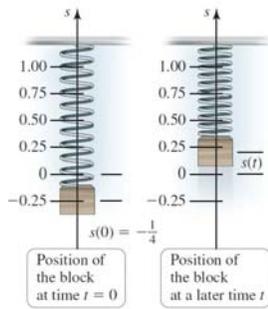

THEOREM 6.1   Position from Velocity

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x)\, dx.$$

position at time t initial position   displacement over $[0, t]$

Figure 6.4

1.00
0.75
0.50
0.25
0
-0.25

Position of the block at time $t = 0$

$s(0) = -\frac{1}{4}$

1.00
0.75
0.50
0.25
0
-0.25

$s(t)$

Position of the block at a later time t

2

Figure 6.5 (a)

Velocity function

$v(t) = \frac{1}{4} \sin t$

0.25

-0.25

π 2π 3π

Figure 6.5 (b)

Position function

$s(t) = -\frac{1}{4} \cos t$

0.25

-0.25

π 2π 3π

Figure 6.6

$\int_0^{19} 80\, dt$

$v(t) = 80$

$\int_{19}^{21} (783 - 37t)\, dt$

$\int_{21}^{40} 6\, dt$

$v(t) = 783 - 37t$

$v(t) = 6$

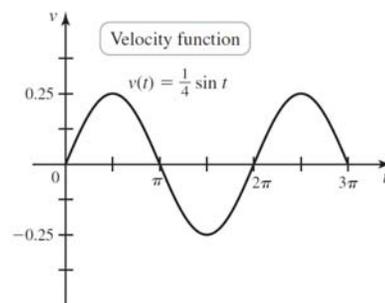Distance traveled $= \int_0^{40} |v(t)|\, dt = \int_0^{19} 80\, dt + \int_{19}^{21} (783 - 37t)\, dt + \int_{21}^{40} 6\, dt$
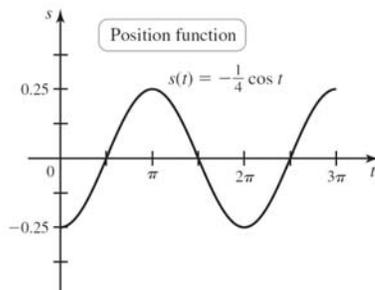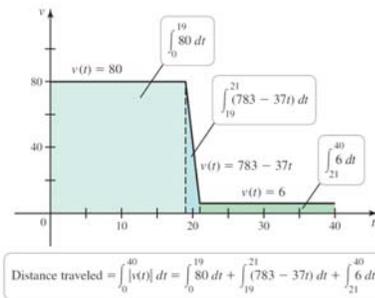
THEOREM 6.2 **Velocity from Acceleration**

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

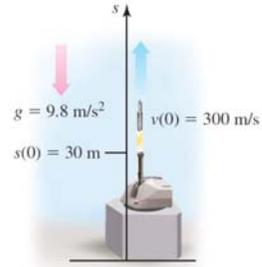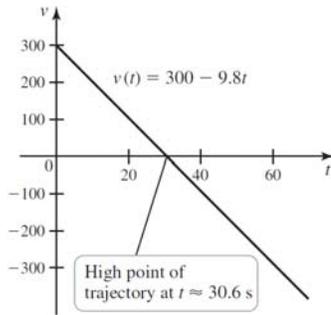$$v(t) = v(0) + \int_0^t a(x)\, dx.$$

Figure 6.7

$g = 9.8 \text{ m/s}^2$

$v(0) = 300$ m/s

$s(0) = 30$ m

Figure 6.8

$v(t) = 300 - 9.8t$

High point of trajectory at $t \approx 30.6$ s

3

THEOREM 6.3 **Net Change and Future Value**

Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b$ is

$$\underbrace{Q(b) - Q(a)}_{\text{net change in } Q} = \int_a^b Q'(t)\, dt.$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\, dx.$$

Table 6.1

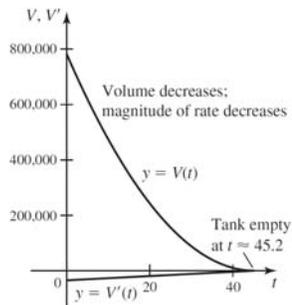| Velocity–Displacement Problems	General Problems
Position $s(t)$	Quantity $Q(t)$ (such as volume or population size)
Velocity: $s'(t) = v(t)$	Rate of change: $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\, dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\, dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\, dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\, dx$

Figure 6.9

R

H

radius r
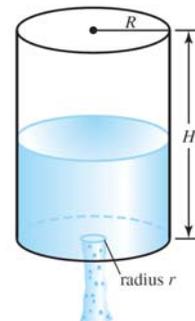
Figure 6.10

6.2

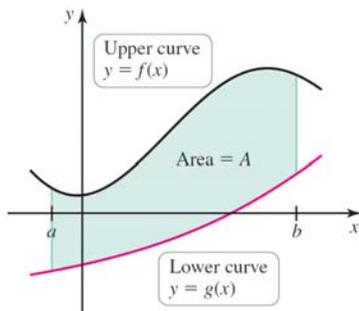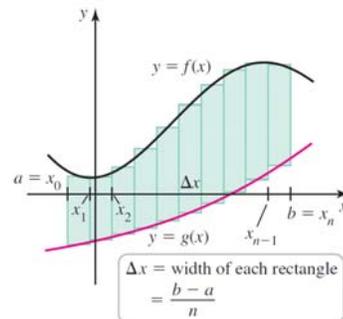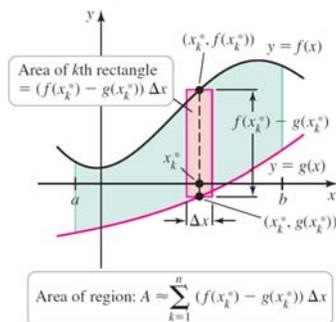Regions Between Curves

Figure 6.11

4

Figure 6.12

Figure 6.13

DEFINITION Area of a Region Between Two Curves

Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

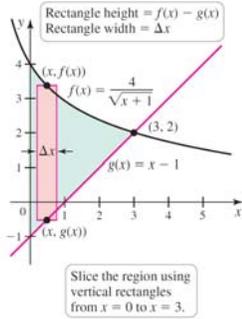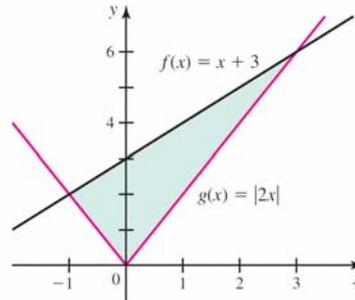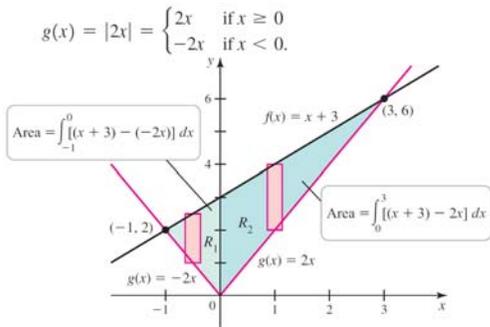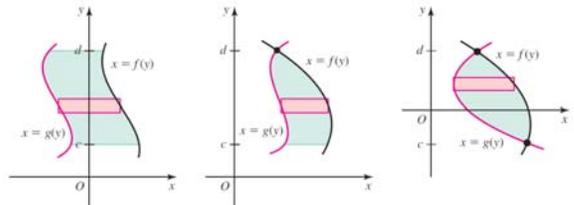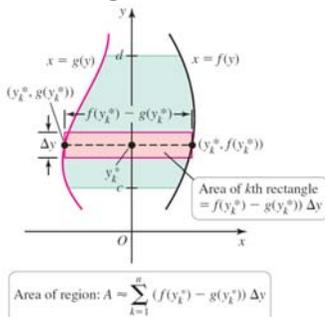$$A = \int_a^b (f(x) - g(x))\, dx.$$

Figure 6.14

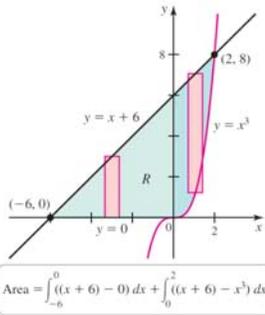Rectangle height $= f(x) - g(x)$
Rectangle width $= \Delta x$

$(x, f(x))$

$f(x) = \dfrac{4}{\sqrt{x+1}}$

$(3, 2)$

Δx

$g(x) = x - 1$

$(x, g(x))$

Slice the region using
vertical rectangles
from $x = 0$ to $x = 3$.

Figure 6.15 (a)

$f(x) = x + 3$

$g(x) = |2x|$

Figure 6.15 (b)

$g(x) = |2x| = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$

$\text{Area} = \displaystyle\int_{-1}^{0} [(x+3) - (-2x)]\, dx$

$f(x) = x + 3$

$(3, 6)$

$(-1, 2)$

R_2

$\text{Area} = \displaystyle\int_{0}^{3} [(x+3) - 2x]\, dx$

R_1

$g(x) = -2x$

$g(x) = 2x$

5

Figure 6.16

$x = f(y)$

$x = g(y)$

$x = f(y)$

$x = g(y)$

$x = f(y)$

$x = g(y)$

Figure 6.17

$x = g(y)$

$x = f(y)$

$(y_k^*, g(y_k^*))$

$f(y_k^*) - g(y_k^*)$

Δy

$(y_k^*, f(y_k^*))$

y_k^*

Area of kth rectangle
$= f(y_k^*) - g(y_k^*))\, \Delta y$

Area of region: $A \approx \displaystyle\sum_{k=1}^{n} (f(y_k^*) - g(y_k^*))\, \Delta y$

DEFINITION Area of a Region Between Two Curves with Respect to y

Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is
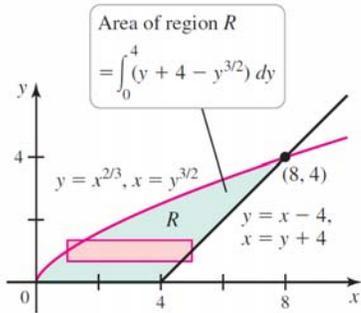
$$A = \int_{c}^{d} (f(y) - g(y))\, dy.$$

Figure 6.18

$y = x + 6$

$y = x^3$

(2, 8)

$y = x$

(−6, 0)

R

$y = 0$

$$\text{Area} = \int_{-6}^{0} ((x + 6) - 0)\, dx + \int_{0}^{2} ((x + 6) - x^3)\, dx$$

Figure 6.19

$$\text{Area} = \int_{0}^{8} (y^{1/3} - (y - 6))\, dy$$

$x = y^{1/3}$

(2, 8)

Rectangle length
$= y^{1/3} - (y - 6)$
Rectangle width
$= \Delta y$

$x = y - 6$

R

(−6, 0)

$y = 0$

Slice the region using horizontal rectangles from $y = 0$ to $y = 8$.

Figure 6.20

Area of region R
$$= \int_{0}^{4} (y + 4 - y^{3/2})\, dy$$

$y = x^{2/3}, x = y^{3/2}$

(8, 4)

R

$y = x - 4,$
$x = y + 4$

6

Figure 6.21

$y = x^{2/3}$

R

$y = x - 4$

Area of region R = (area of region R_1) − (area of region R_2)

R_1

R_2
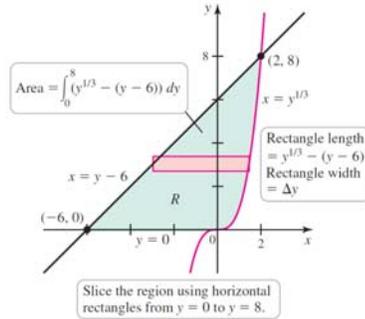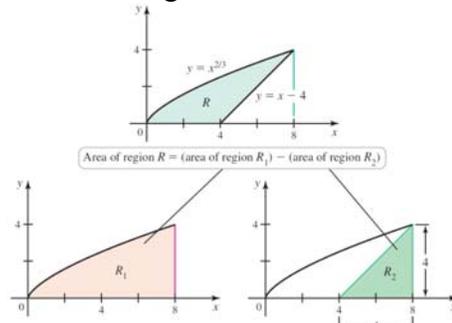
6.3

Volume by Slicing

Figure 6.22

Cross section
with area $A(x)$

a

x

b x

Figure 6.23

Cross-sectional area $= A(x_k^*)$

Δx

Volume of kth slice $\approx A(x_k^*)\,\Delta x$

Figure 6.24

Increase the number of slices.

Volume $= \lim_{n \to \infty} \sum_{k=1}^{n} A(x_k^*)\Delta x$
$= \int_a^b A(x)dx$

$n \to \infty$
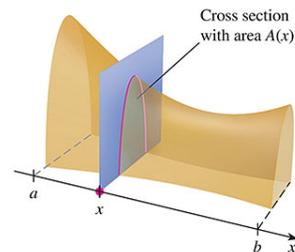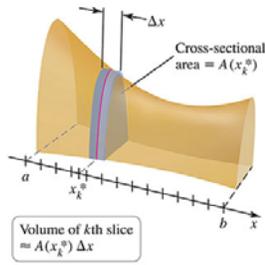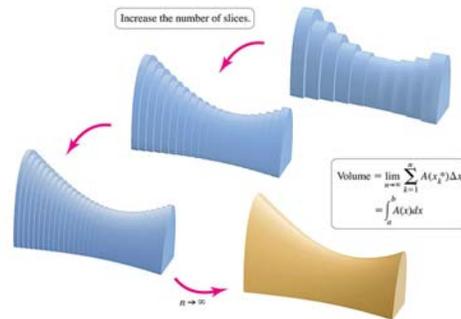
General Slicing Method

Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx.$$

7

Figure 6.25

$$r = \frac{1}{2}\left((2 - x^2) - x^2\right) = 1 - x^2.$$

Semicircular slice

Radius of slice
$= \frac{1}{2}((2 - x^2) - x^2)$
$= 1 - x^2$

$y = x^2$

$y = 2 - x^2$

Base of solid

Figure 6.26

$y = f(x)$

R

Figure 6.27

R

Revolving the region R generates a solid of revolution.

Figure 6.28

Cross sections of a solid of revolution are circular disks of radius $f(x)$ and area $\pi f(x)^2$.
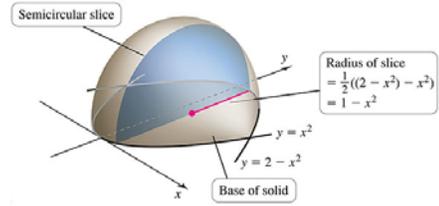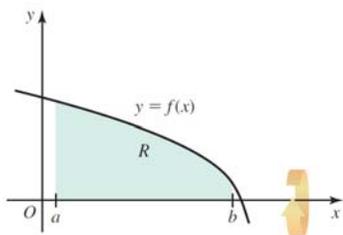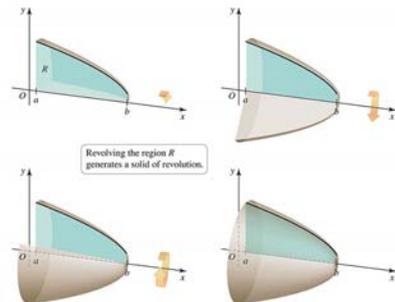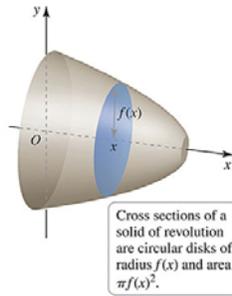
Disk Method About the x-Axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2 \, dx.$$

Figure 6.29 (1 of 2)

$f(x) = (x + 1)^2$

Radius $= (x + 1)^2$

Figure 6.29 (2 of 2)

8

Cross-sectional area $= \pi((x + 1)^2)^2$

Figure 6.30

Revolving region R around the x-axis...

... produces a solid with a hole.

$y = f(x)$

$y = g(x)$

Figure 6.31

$(x, f(x))$

$(x, g(x))$

Cross-sectional area $= \pi(f(x)^2 - g(x)^2)$
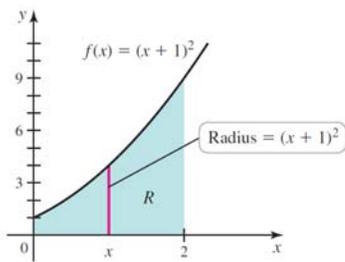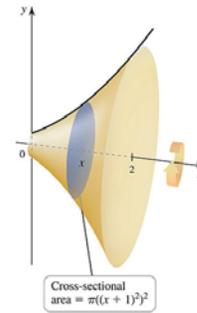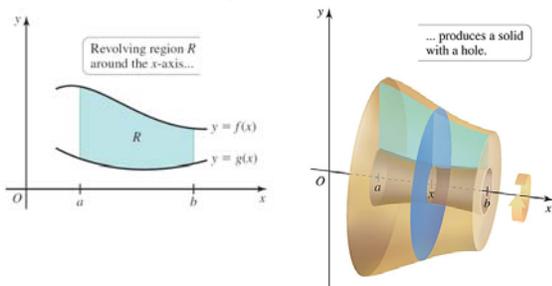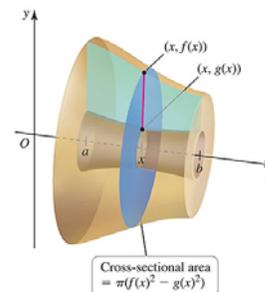
Washer Method About the x-Axis

Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)\, dx.$$

Figure 6.32
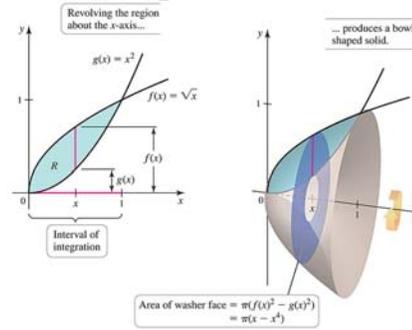
Figure 6.33 (a & b)
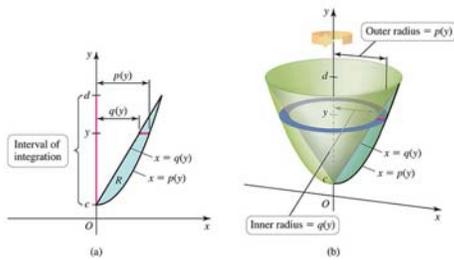
9

Disk and Washer Methods About the y-Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi(p(y)^2 - q(y)^2)\, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\, dy.$$

Figure 6.34
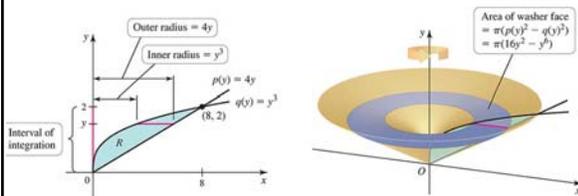
Figure 6.35
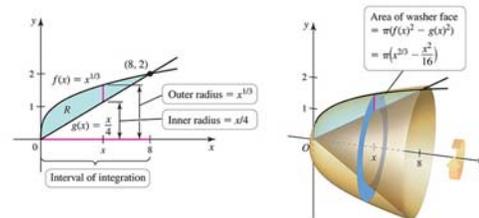
6.4

Volume by Shells

Figure 6.36
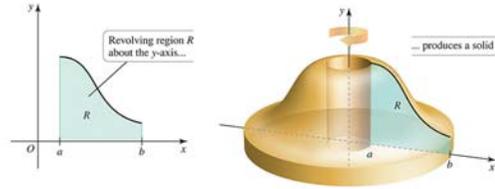
Figure 6.37
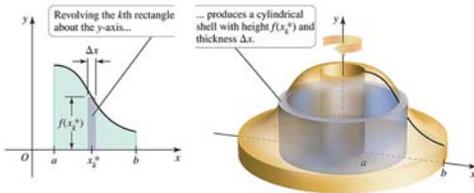
Figure 6.38
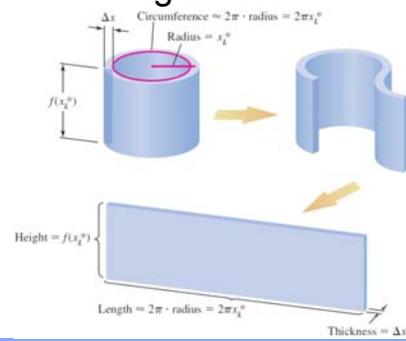
Figure 6.39
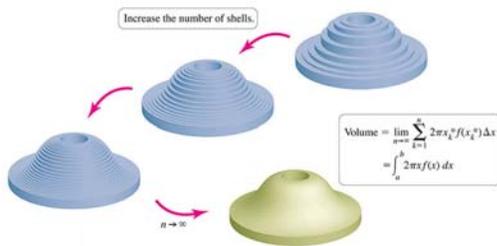
Figure 6.40
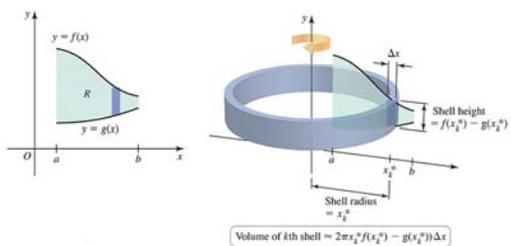
Volume by the Shell Method

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is
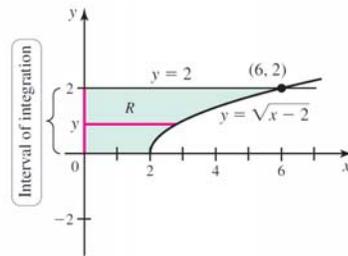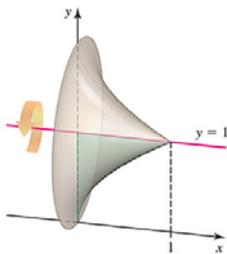
$$V = \int_a^b 2\pi x(f(x) - g(x))\, dx.$$

Figure 6.41

Figure 6.42

1
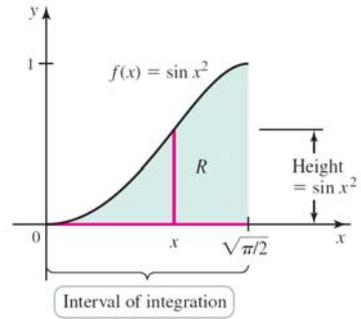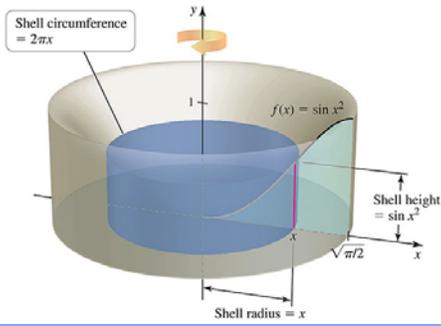
Figure 6.43
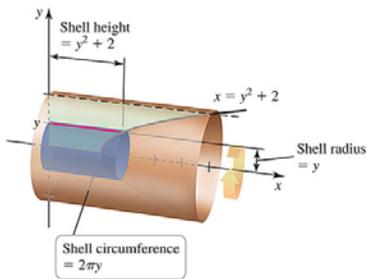
Figure 6.44

Figure 6.45 (a & b)
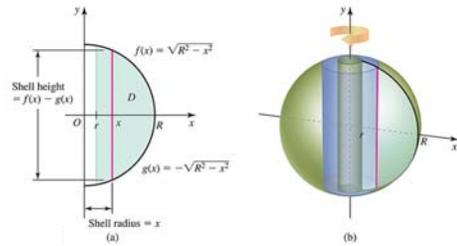
Figure 6.46 (a)
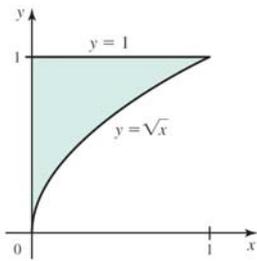
Figure 6.46 (b)
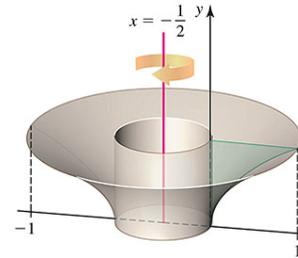
Figure 6.46 (c)

Figure 6.47
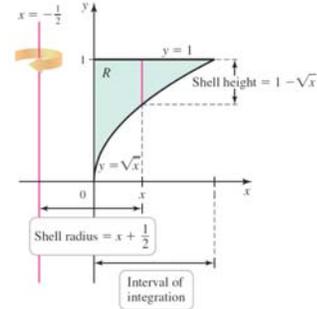
Figure 6.48
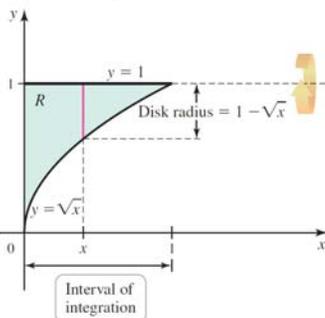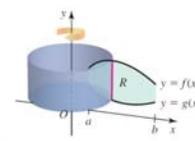

Summary: Disk/Washer and Shell Methods *continued…*

Summary: Disk/Washer and Shell Methods

Integration with respect to y
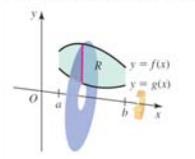
Disk/washer method about the y-axis
Disks/washers are *perpendicular* to the y-axis.

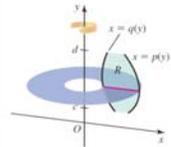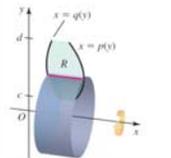$$\int_c^d \pi(p(y)^2 - q(y)^2)\, dy$$

Shell method about the x-axis
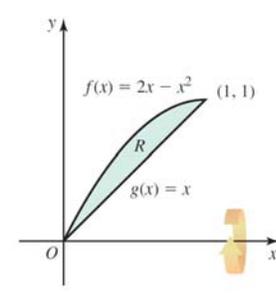Shells are *parallel* to the x-axis.

$$\int_c^d 2\pi y(p(y) - q(y))\, dy$$

Figure 6.49

$f(x) = 2x - x^2$ (1, 1)

R

$g(x) = x$

Figure 6.50

$y = 2x - x^2$ (1, 1)

$y = x$

$g(x)$ $f(x)$
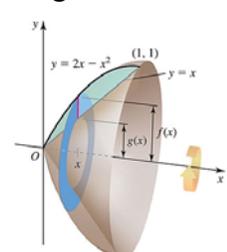
(Outer radius)^2 = $(2x - x^2)^2$
(Inner radius)^2 = x^2

Figure 6.51

$x = 1 - \sqrt{1 - y}$
(1, 1)

$x = y$
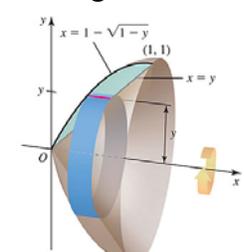
Shell height = $y - (1 - \sqrt{1 - y})$
Shell radius = y

6.5

Length of Curves

Figure 6.52

The length of the red polygonal line from $(a, f(a))$ to $(b, f(b))$ approximates L.
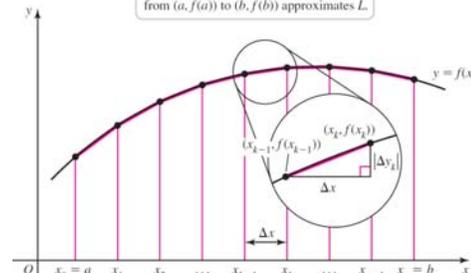
$y = f(x)$

$(x_k, f(x_k))$

$(x_{k-1}, f(x_{k-1}))$

$|\Delta y_k|$

Δx

$x_0 = a$ x_1 x_2 $\cdots$ x_{k-1} x_k $\cdots$ x_{n-1} $x_n = b$

DEFINITION Arc Length for $y = f(x)$

Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

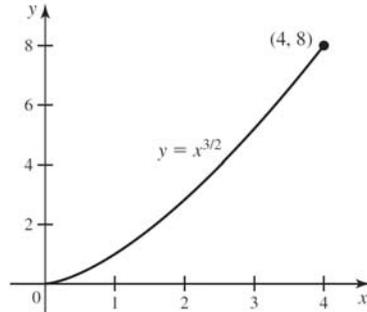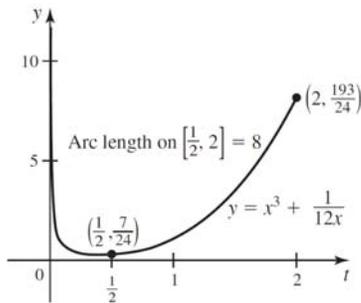$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Figure 6.53

$(4, 8)$

$y = x^{3/2}$

Figure 6.54

$\left(2, \frac{193}{24}\right)$

Arc length on $\left[\frac{1}{2}, 2\right] = 8$

$y = x^3 + \dfrac{1}{12x}$

$\left(\frac{1}{2}, \frac{7}{24}\right)$
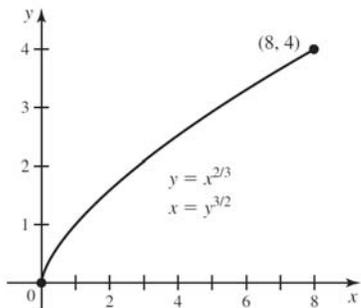

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DEFINITION Arc Length for $x = g(y)$

Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is
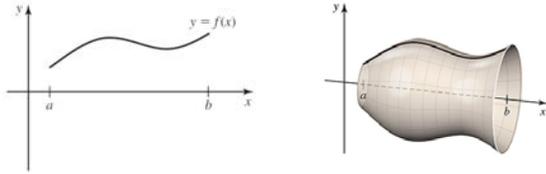
$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Figure 6.55

$(8, 4)$

$y = x^{2/3}$

$x = y^{3/2}$

6.6

Surface Area

Figure 6.56

Figure 6.57

$$\text{Surface area} = \pi r \ell = \pi r \sqrt{r^2 + h^2}$$

Figure 6.58

| Surface area of large cone | − | Surface area of small cone | = | Surface area of frustrum |

Surface area S_b Surface area S_a Surface area $S = S_a - S_b$

Figure 6.59

$y = f(x)$

1

Figure 6.60

$$|\Delta y_k| = |f(x_k) - f(x_{k-1})|$$
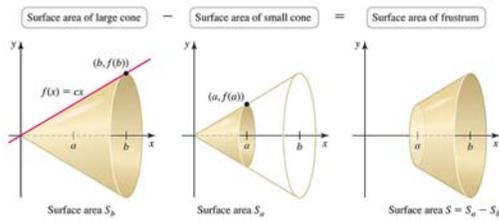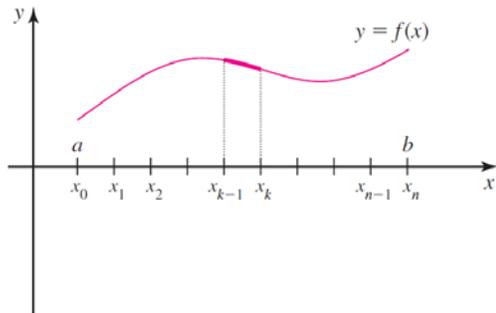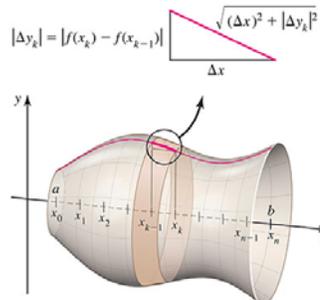
$$\sqrt{(\Delta x)^2 + |\Delta y_k|^2}$$

Δx
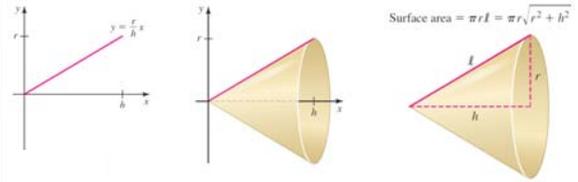
DEFINITION   **Area of a Surface of Revolution**

Let f be differentiable and positive on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is

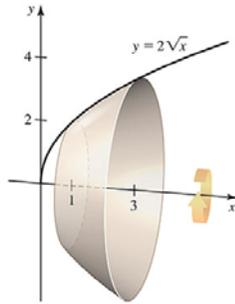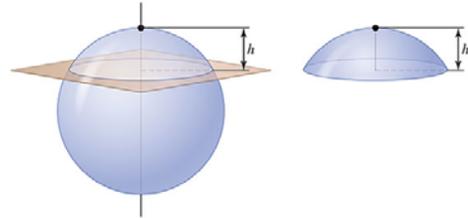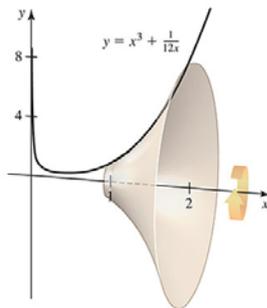$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx.$$

Figure 6.61

$y = 2\sqrt{x}$

Figure 6.62

h

h

Figure 6.63

$y = x^3 + \frac{1}{12x}$

Figure 6.64

$y = \ln\left(\frac{x + \sqrt{x^2 - 1}}{2}\right)$

$\ln 2$

$\frac{5}{4}$ $\frac{17}{8}$
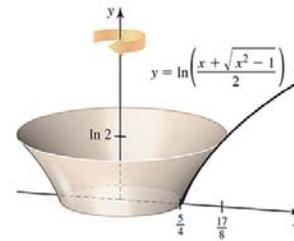
6.7

Physical Applications

Figure 6.65

$x = a$ $x = b$

Figure 6.66

Mass of kth subinterval:
$m_k \approx \rho(x_k^*)\Delta x$

$x = a$ x_k^* $x = b$
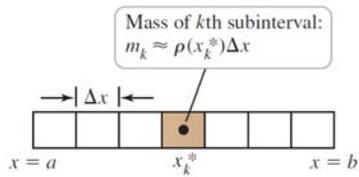
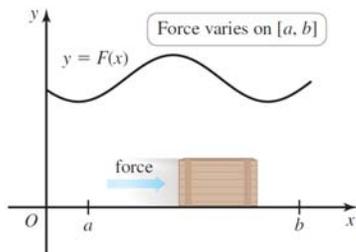**DEFINITION Mass of a One-Dimensional Object**

Suppose a thin bar or wire is represented by a line segment on the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The **mass** of the object is

$$m = \int_a^b \rho(x)\, dx.$$

Figure 6.67

Force varies on $[a, b]$

$y = F(x)$

force

O a b x

DEFINITION Work

The work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is
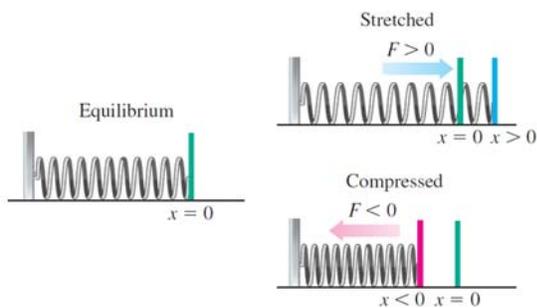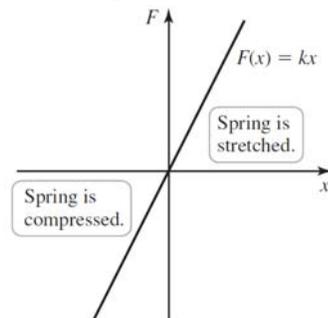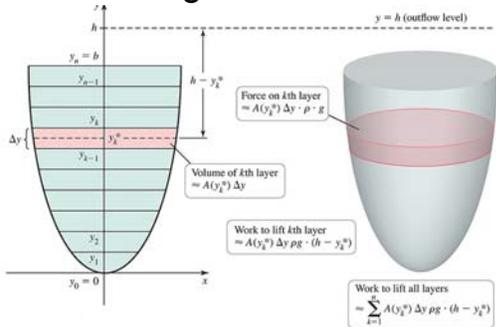
$$W = \int_a^b F(x)\, dx.$$

Figure 6.68

Stretched
$F > 0$

Equilibrium

$x = 0$ $x > 0$

$x = 0$

Compressed
$F < 0$

$x < 0$ $x = 0$

Figure 6.69

F

$F(x) = kx$

Spring is stretched.

x

Spring is compressed.

Figure 6.70

PROCEDURE   **Solving Lifting Problems**

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.

2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is
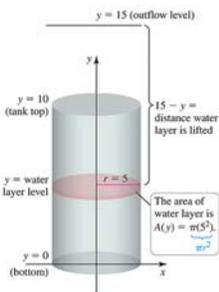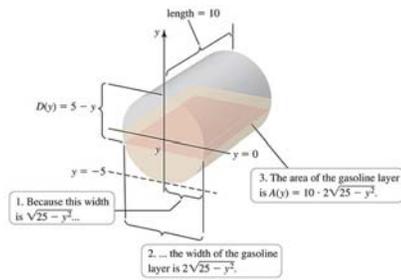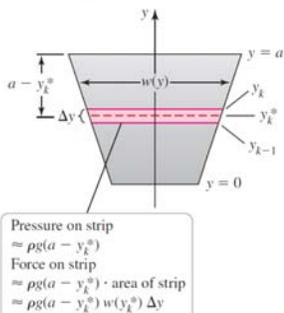
$$W = \int_a^b \rho g A(y) D(y) \, dy.$$

Figure 6.71

Figure 6.72

1

Figure 6.73

PROCEDURE   **Solving Force/Pressure Problems**

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).

2. Find the width function $w(y)$ for each value of y on the face of the dam.

3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a - y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}} \, dy.$$
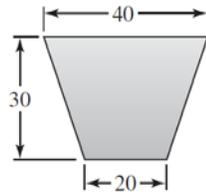
Figure 6.74

Figure 6.75

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