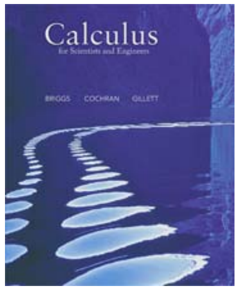
Chapter 7

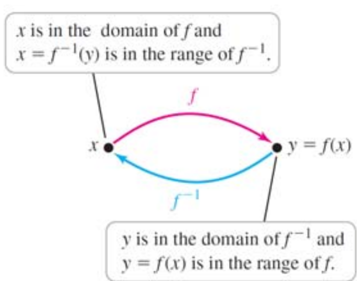


Logarithmic and Exponential Functions

7.1

Inverse Functions

Figure 7.1



1

DEFINITION Inverse Function

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$ (Figure 7.1).

DEFINITION One-to-One Functions and the Horizontal Line Test

A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D. The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once (Figure 7.2).

Figure 7.2

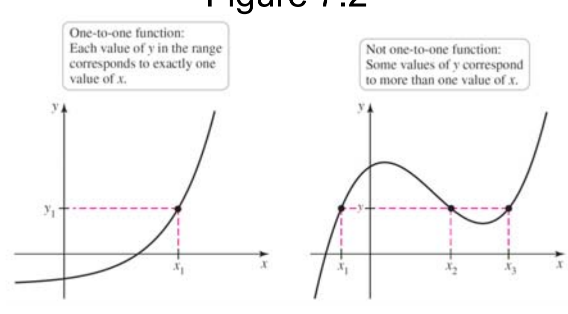


Figure 7.3

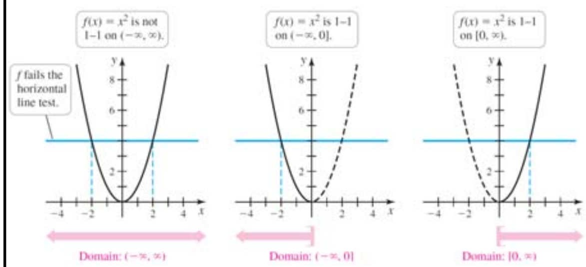


Figure 7.4

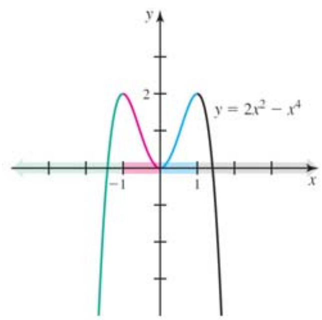
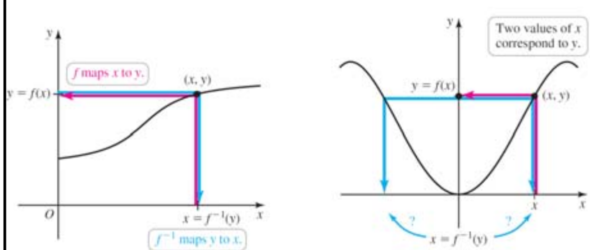


Figure 7.5 (a & b)



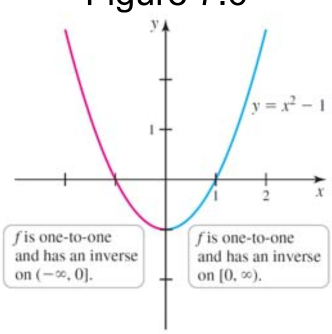
2

THEOREM 7.1 Existence of Inverse Functions
Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y,$$

where x is in D and y is in R.

Figure 7.6



PROCEDURE Finding an Inverse Function
Suppose f is one-to-one on an interval I. To find f^{-1}:

1. Solve $y = f(x)$ for x. If necessary, restrict the resulting function so that x lies in I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Figure 7.7

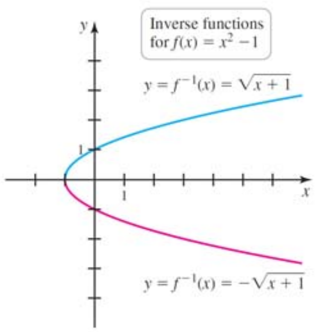


Figure 7.8

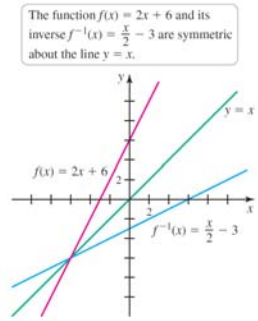


Figure 7.9

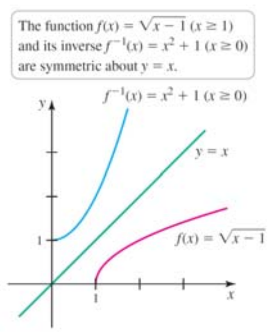
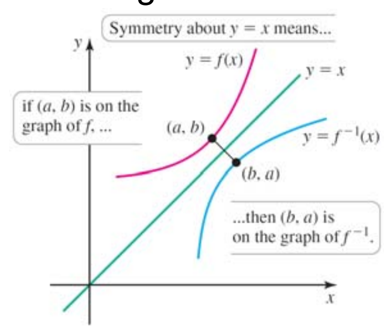


Figure 7.10



THEOREM 7.2 Continuity of Inverse Functions
If a continuous function f has an inverse on an interval I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Figure 7.11

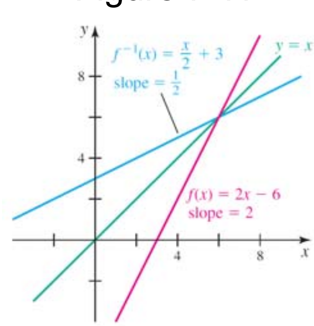
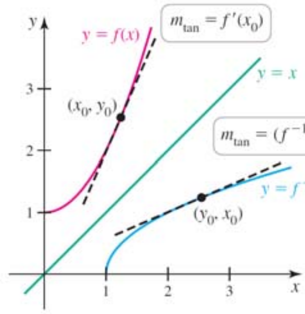


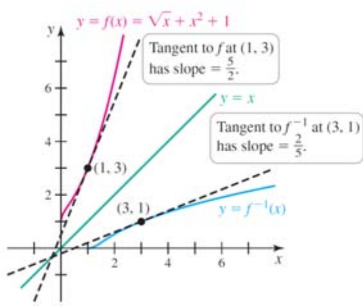
Figure 7.12



THEOREM 7.3 Derivative of the Inverse Function
Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \quad \text{where} \quad y_0 = f(x_0).$$

Figure 7.13



7.2

The Natural Logarithmic and Exponential Functions

DEFINITION The Natural Logarithm
The **natural logarithm** of a number $x > 0$, denoted $\ln x$, is defined as

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

Figure 7.14 (a & b)

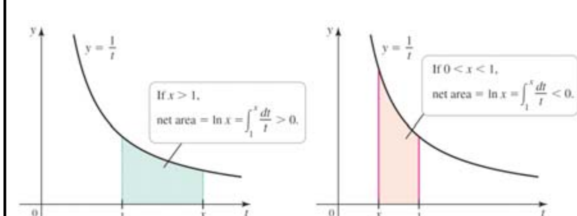


Figure 7.15

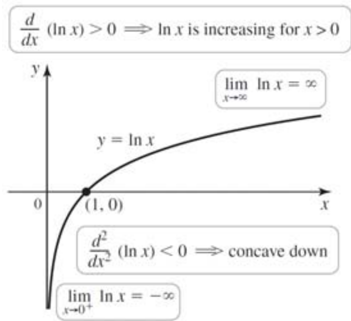
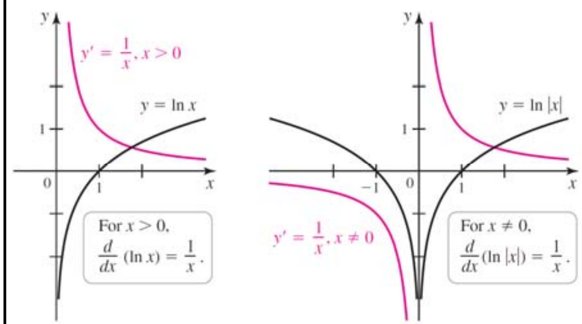


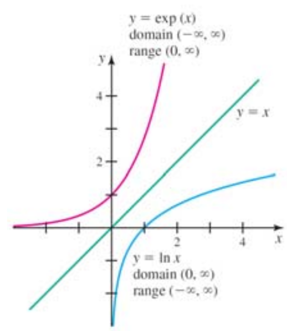
Figure 7.16 (a & b)



THEOREM 7.4 Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln(xy) = \ln x + \ln y$, for $x > 0, y > 0$
3. $\ln(x/y) = \ln x - \ln y$, for $x > 0, y > 0$
4. $\ln x^p = p \ln x$, for $x > 0$ and p a rational number
5. $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$, for $x \neq 0$
6. $\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}$, for $u(x) \neq 0$
7. $\int \frac{1}{x}\,dx = \ln |x| + C$

Figure 7.17

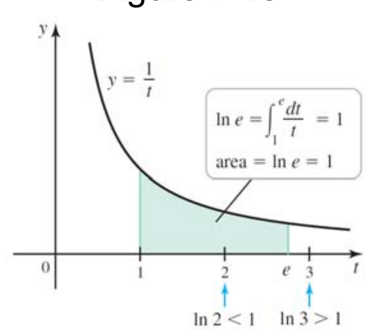


DEFINITION The Number e

The number e is the real number that satisfies

$$\ln e = \int_1^e \frac{dt}{t} = 1.$$

Figure 7.18



DEFINITION The Exponential Function

For real numbers x, $y = e^x = \exp(x)$, where $x = \ln y$.

THEOREM 7.5 Properties of e^x

The exponential function e^x satisfies the following properties, all of which follow from the integral definition of $\ln x$. Let x and y be real numbers.

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = e^x / e^y$
3. $(e^x)^y = e^{xy}$, for y a rational number
4. $\ln(e^x) = x$, for all x
5. $e^{\ln x} = x$, for $x > 0$

DEFINITION Exponential Functions with General Bases

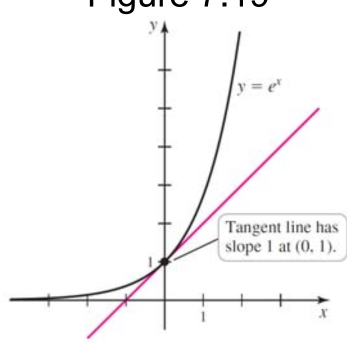
Let b be a positive real number with $b \neq 1$. Then for all real x,

$$b^x = e^{x \ln b}.$$

Table 7.1

h	$(1+h)^{1/h}$	h	$(1+h)^{1/h}$
10^{-1}	2.593742	-10^{-1}	2.867972
10^{-2}	2.704814	-10^{-2}	2.731999
10^{-3}	2.716924	-10^{-3}	2.719642
10^{-4}	2.718146	-10^{-4}	2.718418
10^{-5}	2.718268	-10^{-5}	2.718295
10^{-6}	2.718280	-10^{-6}	2.718283
10^{-7}	2.718282	-10^{-7}	2.718282

Figure 7.19

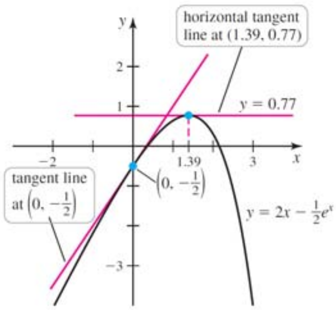


THEOREM 7.6 Derivative and Integral of the Exponential Function

For real numbers x,

$$\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)}u'(x) \quad \text{and} \quad \int e^x\,dx = e^x + C.$$

Figure 7.20



7.3

Logarithmic and Exponential Functions with Other Bases

Figure 7.21

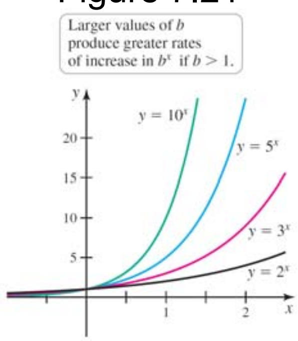


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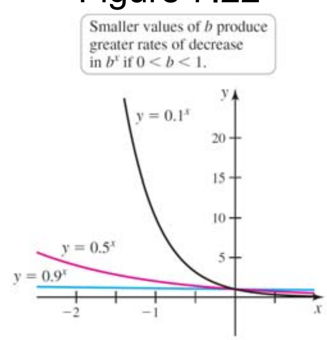
Figure 7.22



DEFINITION Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base b**, denoted $\log_b x$, is the inverse of the exponential function b^x.

Inverse Relations for Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relations hold.

I1. $b^{\log_b x} = x$, for $x > 0$

I2. $\log_b b^x = x$, for all x

Figure 7.23

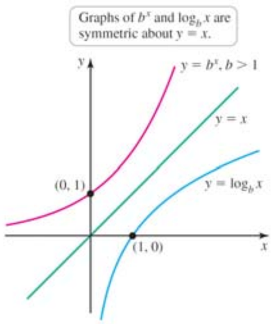
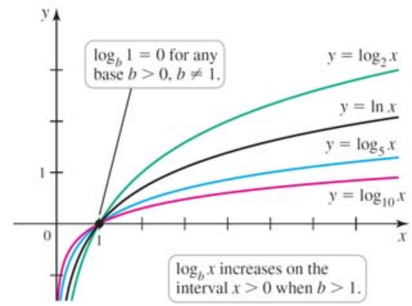


Figure 7.24



THEOREM 7.7 Derivative of b^x
If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

Figure 7.25

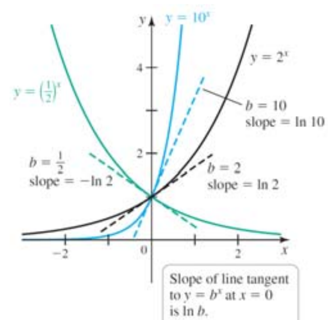
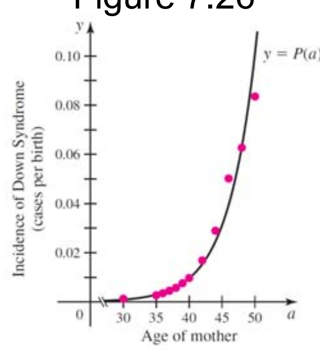


Table 7.2

Mother's Age	Incidence of Down Syndrome	Decimal Equivalent
30	1 in 900	0.00111
35	1 in 400	0.00250
36	1 in 300	0.00333
37	1 in 230	0.00435
38	1 in 180	0.00556
39	1 in 135	0.00741
40	1 in 105	0.00952
42	1 in 60	0.01667
44	1 in 35	0.02875
46	1 in 20	0.05000
48	1 in 16	0.06250
49	1 in 12	0.08333

Source: E.G. Hook and A. Lindsjo, *Down Syndrome in Live Births by Single Year Maternal Age*

Figure 7.26



THEOREM 7.8 Indefinite integral of b^x

For $b > 0$ and $b \neq 1$, $\int b^x \, dx = \frac{1}{\ln b} b^x + C$.

THEOREM 7.9 General Power Rule

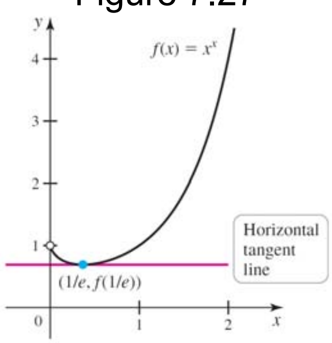
For real numbers p and for $x > 0$,

$$\frac{d}{dx}(x^p) = px^{p-1}.$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \cdot u'(x).$$

Figure 7.27



THEOREM 7.10 Derivative of $\log_b x$

If $b > 0$ with $b \neq 1$, then

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \quad \text{for } x > 0 \quad \text{and} \quad \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \quad \text{for } x \neq 0.$$

7.4

Exponential Models

Figure 7.28

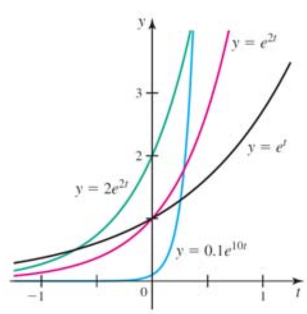
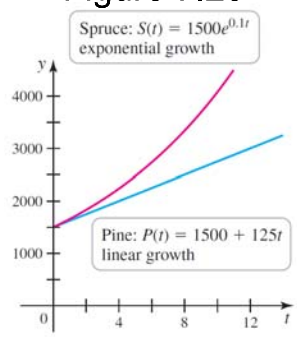


Figure 7.29



Exponential Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0e^{kt}$. The initial value of y at $t = 0$ is $y(0) = y_0$ and the **rate constant** $k > 0$ determines the rate of growth. Exponential growth is characterized by a constant relative growth rate.

DEFINITION Doubling Time

The quantity described by the function $y(t) = y_0e^{kt}$, for $k > 0$, has a constant **doubling time** of $T_2 = \frac{\ln 2}{k}$, with the same units as t.

Figure 7.30

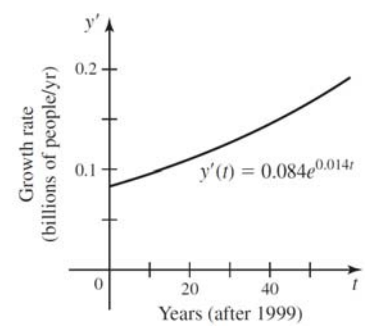


Figure 7.31

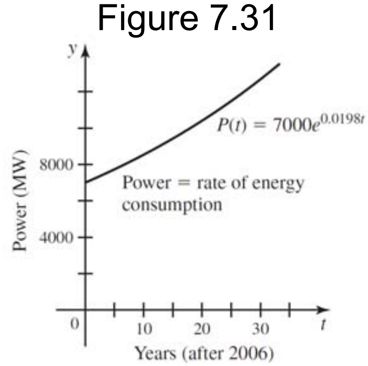
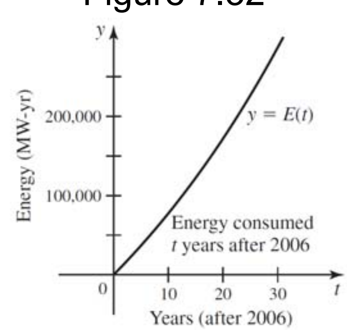


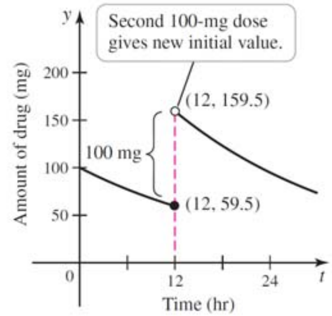
Figure 7.32



Exponential Decay Functions

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$. The initial value of y is $y(0) = y_0$, and the rate constant $k > 0$ determines the rate of decay. Exponential decay is characterized by a constant relative decay rate. The constant **half-life** is $T_{1/2} = \frac{\ln 2}{k}$, with the same units as t.

Figure 7.33



7.5

Inverse Trigonometric Functions

1

Figure 7.34

Infinitely many values of x satisfy $\sin x = \frac{1}{2}$.

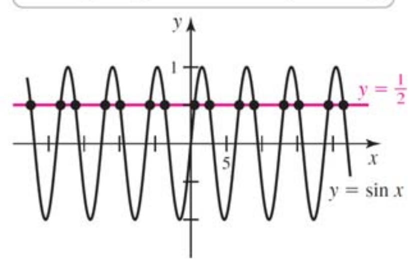


Figure 7.35 (a)

Restrict the domain of $y = \sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

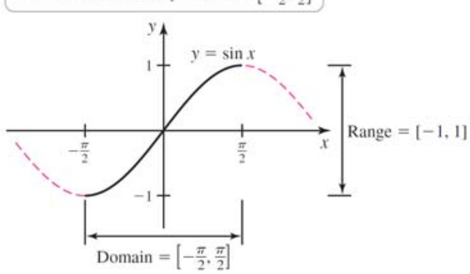
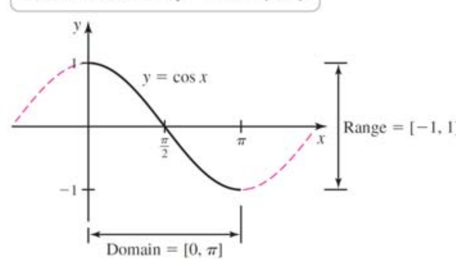


Figure 7.35 (b)

Restrict the domain of $y = \cos x$ to $[0, \pi]$.



DEFINITION Inverse Sine and Cosine

$y = \sin^{-1} x$ is the value of y such that $x = \sin y$, where $-\pi/2 \le y \le \pi/2$.
$y = \cos^{-1} x$ is the value of y such that $x = \cos y$, where $0 \le y \le \pi$.
The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $\{x: -1 \le x \le 1\}$.

Figure 7.36

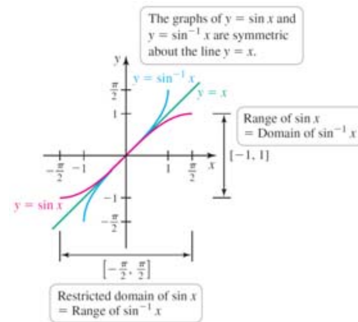


Figure 7.37

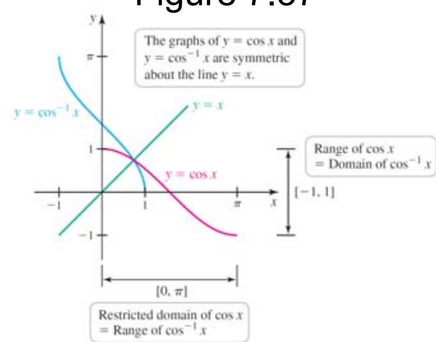


Figure 7.38

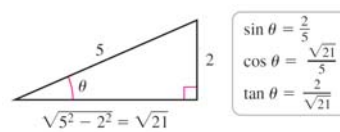


Figure 7.39

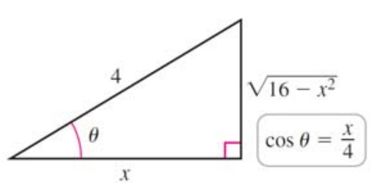
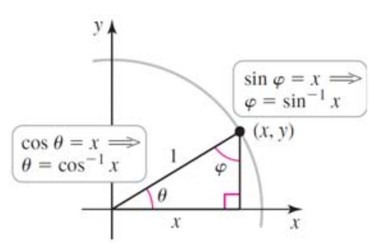


Figure 7.40



DEFINITION Other Inverse Trigonometric Functions

$y = \tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\pi/2 < y < \pi/2$.

$y = \cot^{-1} x$ is the value of y such that $x = \cot y$, where $0 < y < \pi$.

The domain of both $\tan^{-1} x$ and $\cot^{-1} x$ is $\{x: -\infty < x < \infty\}$.

$y = \sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 \le y \le \pi$, with $y \ne \pi/2$.

$y = \csc^{-1} x$ is the value of y such that $x = \csc y$, where $-\pi/2 \le y \le \pi/2$, with $y \ne 0$.

The domain of both $\sec^{-1} x$ and $\csc^{-1} x$ is $\{x: |x| \ge 1\}$.

Figure 7.41

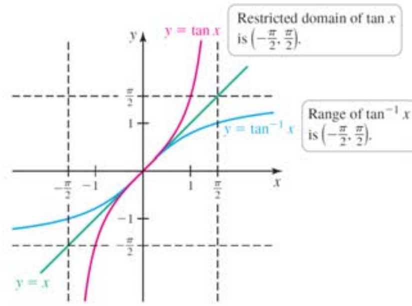


Figure 7.42

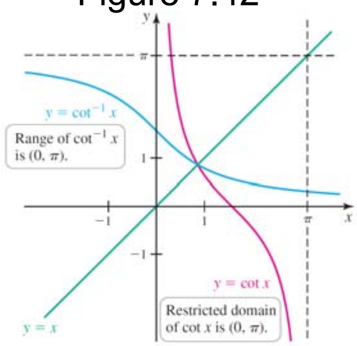


Figure 7.43

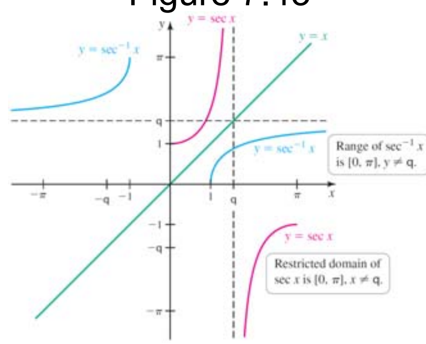


Figure 7.44

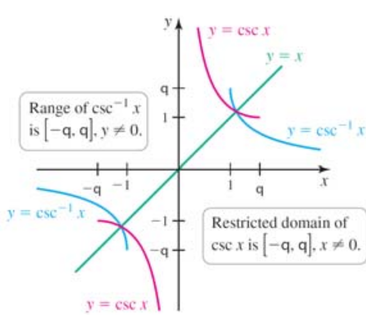


Figure 7.45

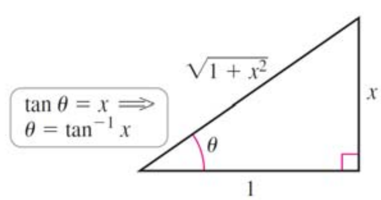


Figure 7.46

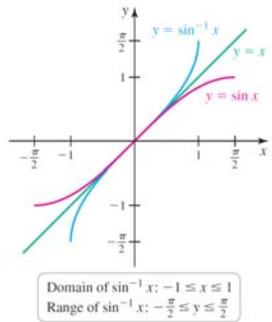
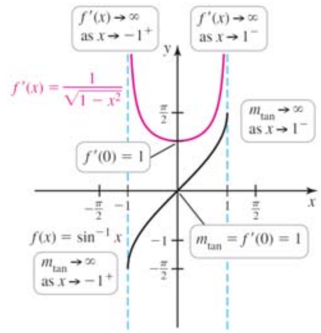


Figure 7.47



THEOREM 7.11 **Derivative of Inverse Sine**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1$$

14

Figure 7.48

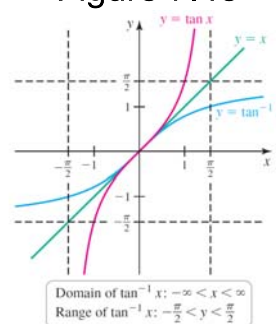


Figure 7.49

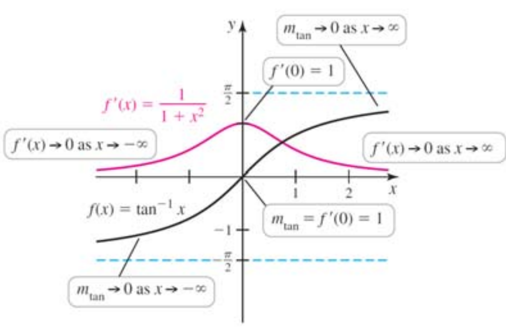
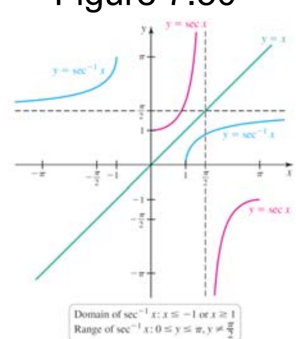


Figure 7.50



THEOREM 7.12 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad \text{for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad \text{for } |x| > 1$$

Figure 7.51

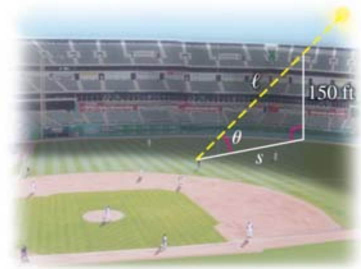
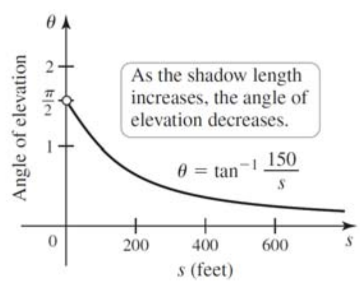


Figure 7.52



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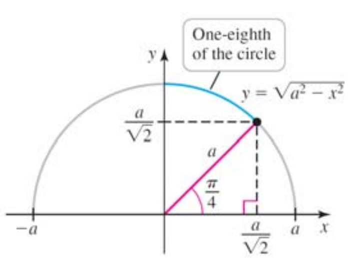
THEOREM 7.13 Integrals Involving Inverse Trigonometric Functions

1. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
2. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
3. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Figure 7.53



7.6

L'Hôpital's Rule and Growth Rates of Functions

PROCEDURE **Indeterminate forms 1^∞, 0^0, and ∞^0**

Assume $\lim_{x \to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{x \to a} g(x) \ln f(x)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by l'Hôpital's Rule.
2. Then $\lim_{x \to a} f(x)^{g(x)} = e^L$.

Figure 7.54

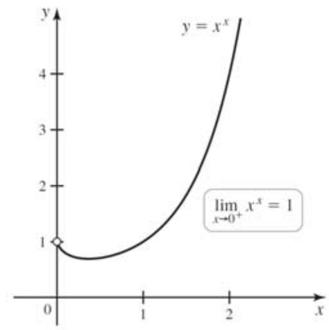
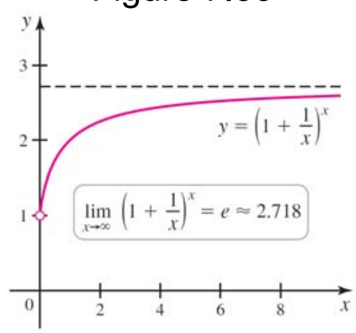


Figure 7.55



DEFINITION **Growth Rates of Functions (as $x \to \infty$)**

Suppose f and g are functions with $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$. Then **$f$ grows faster than g** as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.$$

The functions f and g have **comparable growth rates** if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is nonzero and finite).

Figure 7.56

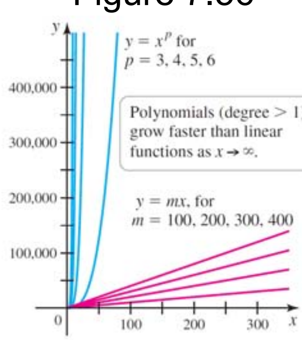
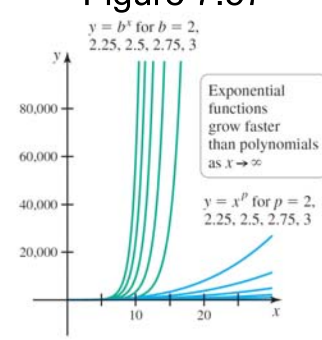


Figure 7.57



THEOREM 7.13 Ranking Growth Rates as $x \to \infty$

Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. With positive real numbers p, q, r, and s and $b > 1$,

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x.$$

7.7

Hyperbolic Functions

Figure 7.58

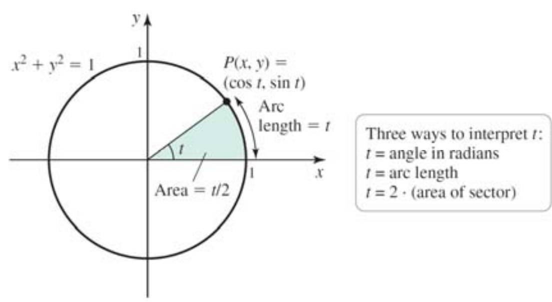
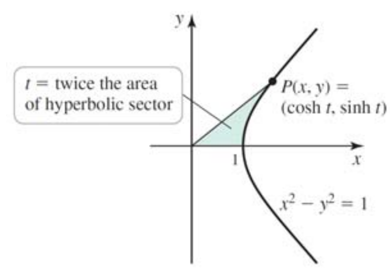


Figure 7.59



DEFINITION Hyperbolic Functions

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$ $\qquad \cosh(-x) = \cosh x$

$1 - \tanh^2 x = \operatorname{sech}^2 x$ $\qquad \sinh(-x) = -\sinh x$

$\coth^2 x - 1 = \operatorname{csch}^2 x$ $\qquad \tanh(-x) = -\tanh x$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh 2x = \cosh^2 x + \sinh^2 x$ $\qquad \sinh 2x = 2 \sinh x \cosh x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$ $\qquad \sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Figure 7.60

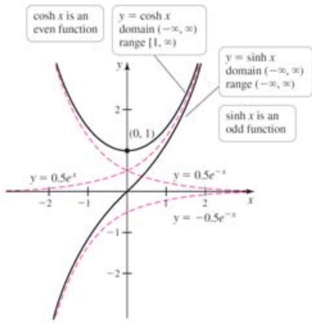
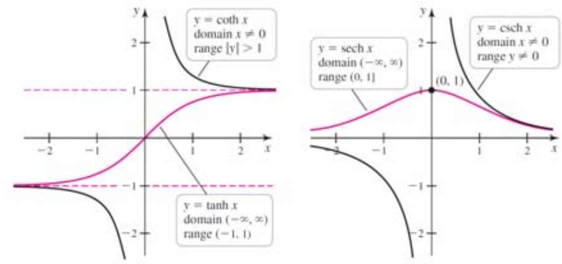


Figure 7.61



THEOREM 7.14 Derivative and Integral Formulas

1. $\frac{d}{dx}(\cosh x) = \sinh x \quad \Rightarrow \quad \int \sinh x\, dx = \cosh x + C$
2. $\frac{d}{dx}(\sinh x) = \cosh x \quad \Rightarrow \quad \int \cosh x\, dx = \sinh x + C$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \Rightarrow \quad \int \operatorname{sech}^2 x\, dx = \tanh x + C$
4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \Rightarrow \quad \int \operatorname{csch}^2 x\, dx = -\coth x + C$
5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \Rightarrow \quad \int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \quad \Rightarrow \quad \int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

1

THEOREM 7.15 Integrals of Hyperbolic Functions

1. $\int \tanh x\, dx = \ln \cosh x + C$
2. $\int \coth x\, dx = \ln|\sinh x| + C$
3. $\int \operatorname{sech} x\, dx = \tan^{-1} \sinh x + C$
4. $\int \operatorname{csch} x\, dx = \ln|\tanh(x/2)| + C$

Figure 7.62 (a & b)

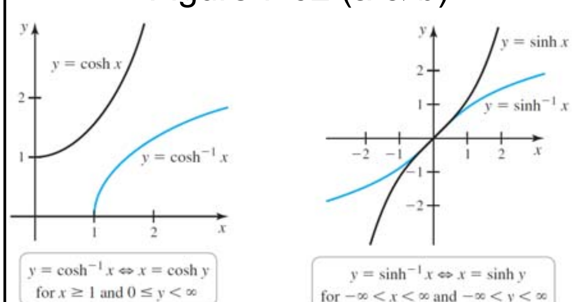


Figure 7.62 (c & d)

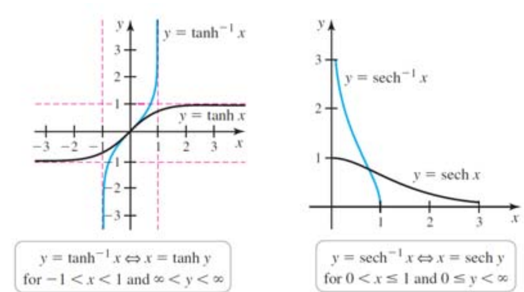
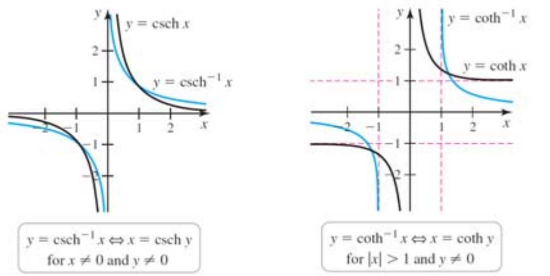


Figure 7.62 (e & f)



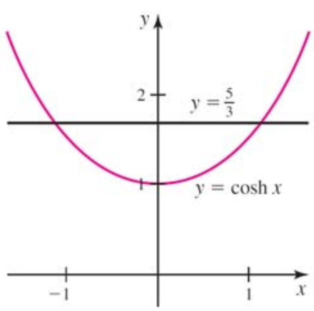
THEOREM 7.16 Inverses of the Hyperbolic Functions Expressed as Logarithms

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})\ (x \geq 1)$ $\quad$ $\text{sech}^{-1} x = \cosh^{-1} \frac{1}{x}\ (0 < x \leq 1)$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ $\quad$ $\text{csch}^{-1} x = \sinh^{-1} \frac{1}{x}\ (x \neq 0)$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ (|x| < 1)$ $\quad$ $\coth^{-1} x = \tanh^{-1} \frac{1}{x}\ (|x| > 1)$

Figure 7.63



THEOREM 7.17 Derivatives of the Inverse Hyperbolic Functions

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}\ (x > 1)$ $\quad$ $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}\ (|x| < 1)$ $\quad$ $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}\ (|x| > 1)$

$\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}\ (0 < x < 1)$ $\quad$ $\frac{d}{dx}(\text{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}}\ (x \neq 0)$

THEOREM 7.18 Integral Formulas

1. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C$, for $x > a$
2. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$, for all x
3. $\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a}\tanh^{-1} \frac{x}{a} + C, & \text{for } |x| < a \\ \frac{1}{a}\coth^{-1} \frac{x}{a} + C, & \text{for } |x| > a \end{cases}$
4. $\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\text{sech}^{-1} \frac{x}{a} + C$, for $0 < x < a$
5. $\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a}\text{csch}^{-1} \frac{|x|}{a} + C$, for $x \neq 0$

Figure 7.64

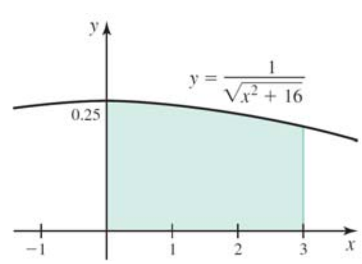


Figure 7.65

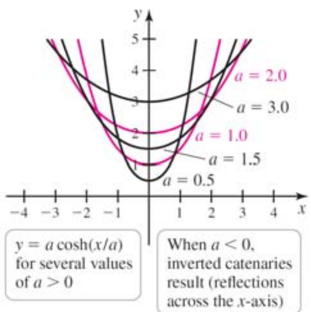


Figure 7.66

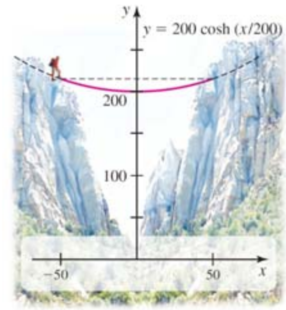


Figure 7.67