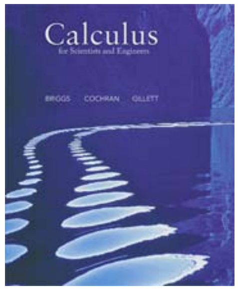
Chapter 8

Integration Techniques



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8.1

Basic Approaches

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Table 8.1 Basic Integration

1. $\int k\,dx = kx + C$, k real
2. $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$ real
3. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
4. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
5. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
6. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
7. $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$
8. $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$
9. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
10. $\int \frac{dx}{x} = \ln|x| + C$
11. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
12. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
13. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

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1

8.2

Integration by Parts

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Integration by Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

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Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

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Integral of ln x

$$\int \ln x \, dx = x \ln x - x + C$$

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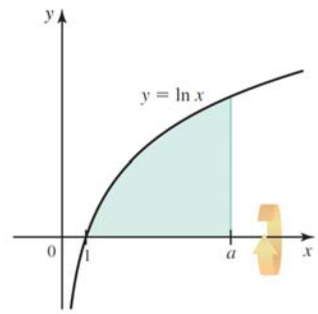
Figure 8.1



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8.3

Trigonometric Integrals

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2

Table 8.2

$\int \sin^m x \cos^n x \, dx$	**Strategy**
m odd, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.
n odd, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.
m and n both even, nonnegative integers	Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

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Reduction Formulas

Assume n is a positive integer.

1. $\displaystyle\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$
2. $\displaystyle\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$
3. $\displaystyle\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \; n \neq 1$
4. $\displaystyle\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \; n \neq 1$

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THEOREM 8.1 Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C \qquad \int \cot x \, dx = \ln |\sin x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \qquad \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

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Table 8.3

$\int \tan^m x \sec^n x\, dx$	**Strategy**
n even	Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
m odd	Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
m even and n odd	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

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8.4

Trigonometric Substitutions

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Figure 8.2

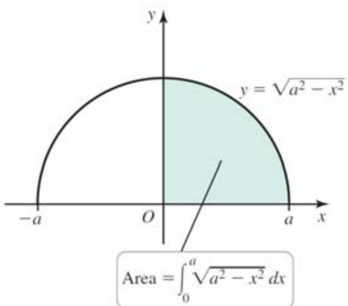

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Figure 8.3

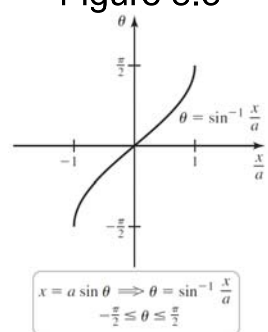

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Figure 8.4

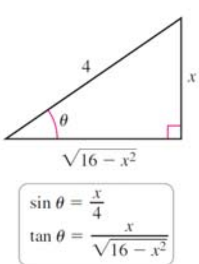

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Figure 8.5

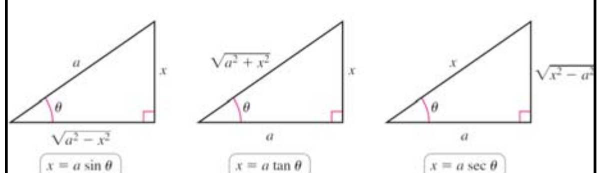

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Table 8.4

The Integral Contains . . .	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\lvert x\rvert \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a \sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2}, \text{ for } x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \text{ for } x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Figure 8.6

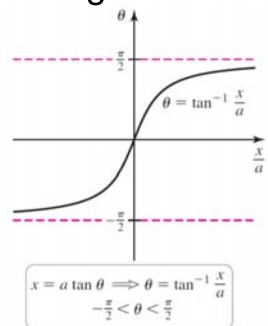


Figure 8.7

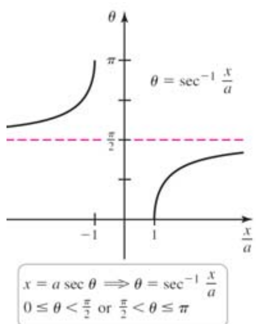


Figure 8.8

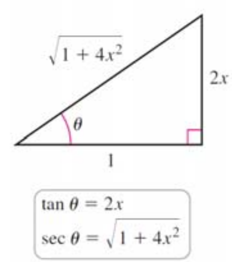
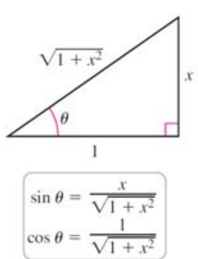


Figure 8.9



8.5

Partial Fractions

Rational function

$$\frac{3x}{x^2 + 2x - 8}$$

method of partial fractions →

Partial fraction decomposition

$$\frac{1}{x-2} + \frac{2}{x+4}$$

Difficult to integrate

$$\int \frac{3x}{x^2 + 2x - 8}\,dx$$

Easy to integrate

$$\int \left(\frac{1}{x-2} + \frac{2}{x+4}\right) dx$$

PROCEDURE Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \ldots, A_n$.

PROCEDURE Partial Fractions for Repeated Linear Factors

Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m},$$

where $A_1, \ldots, A_m$ are constants to be determined.

PROCEDURE Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are unknown coefficients to be determined.

SUMMARY Partial Fraction Decompositions

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x - r}$.
2. **Repeated linear factor** A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m}.$$

3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}.$$

continued…

4. **Repeated irreducible quadratic factor** (See Exercises 83–86.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}.$$

8.6

Other Integration Strategies

Figure 8.10

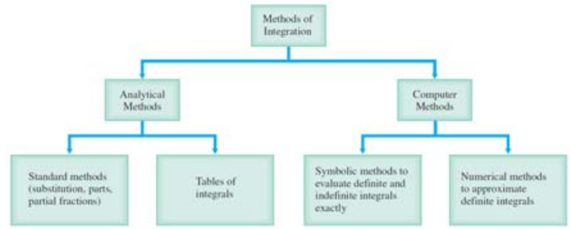
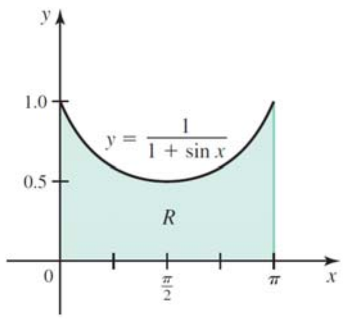


Figure 8.11



8.7

Numerical Integration

DEFINITIONS Absolute and Relative Error

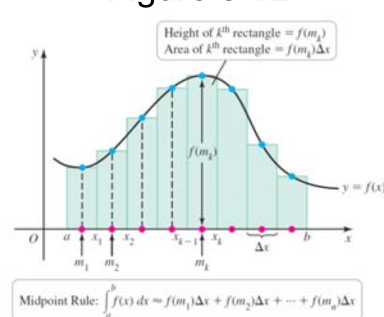
Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0).$$

Figure 8.12



DEFINITION Midpoint Rule

Suppose f is defined and integrable on $[a, b]$. The **Midpoint Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$
$$= \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x,$$

where $\Delta x = (b - a)/n$, $x_k = a + k\Delta x$, and m_k is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, \ldots, n$.

Figure 8.13

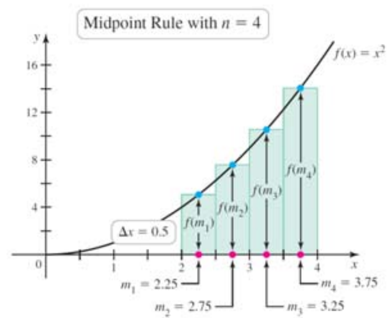
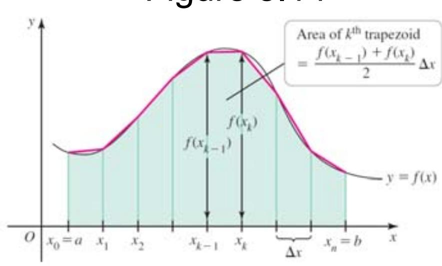


Figure 8.14



Trapezoid Rule: $\int_a^b f(x)dx \approx \left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]\Delta x$

DEFINITION Trapezoid Rule

Suppose f is defined and integrable on $[a, b]$. The **Trapezoid Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$T(n) = \left(\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right)\Delta x,$$

where $\Delta x = (b - a)/n$ and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

Figure 8.15

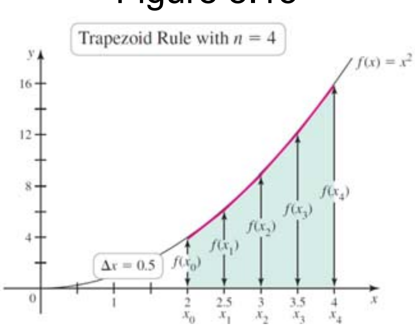


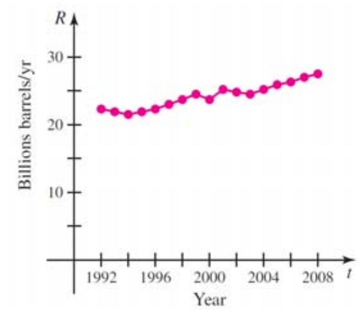
Table 8.5

n	$M(n)$	$T(n)$	Error $M(n)$	Error $T(n)$
4	0.26683456310319	0.25904504019141	0.00259	0.00520
8	0.26489148795740	0.26293980164730	0.000650	0.00130
16	0.26440383609318	0.26391564480235	0.000163	0.000325
32	0.26428180513718	0.26415974044777	0.0000407	0.0000814
64	0.26425129001915	0.26422077279247	0.0000102	0.0000203
128	0.26424366077837	0.26423603140581	0.00000254	0.00000509

Table 8.6

Year	World Oil Production (billions barrels/yr)
1992	22.3
1993	21.9
1994	21.5
1995	21.9
1996	22.3
1997	23.0
1998	23.7
1999	24.5
2000	23.7
2001	25.2
2002	24.8
2003	24.5
2004	25.2
2005	25.9
2006	26.3
2007	27.0
2008	27.5

Figure 8.16



DEFINITION Simpson's Rule

Suppose f is defined and integrable on $[a, b]$. The **Simpson's Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$S(n) = \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]\frac{\Delta x}{3},$$

where n is an even integer, $\Delta x = (b - a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$. Alternatively, if the Trapezoid Rule approximations $T(2n)$ and $T(n)$ are known, then

$$S(2n) = \frac{4T(2n) - T(n)}{3}.$$

Table 8.7

n	$T(n)$	$S(n)$	Error $T(n)$	Error $S(n)$
4	0.25904504019141		0.00520	
8	0.26293980164730	0.26423805546593	0.00130	0.00000306
16	0.26391564480235	0.26424092585404	0.000325	0.000000192
32	0.26415974044777	0.26424110566291	0.0000814	0.0000000120
64	0.26422077279247	0.26424111690738	0.0000203	0.000000000750
128	0.26423603140581	0.26424111761026	0.00000509	0.0000000000469

THEOREM 8.2 Errors in Numerical Integration

Assume that f'' is continuous on the interval $[a, b]$ and that k is a real number such that $|f''(x)| < k$, for all x in $[a, b]$. The absolute errors in approximating the integral $\int_a^b f(x)\,dx$ by the Midpoint Rule and Trapezoid Rule with n subintervals satisfy the inequalities

$$E_M \le \frac{k(b-a)}{24}(\Delta x)^2 \quad \text{and} \quad E_T \le \frac{k(b-a)}{12}(\Delta x)^2,$$

respectively, where $\Delta x = (b - a)/n$.

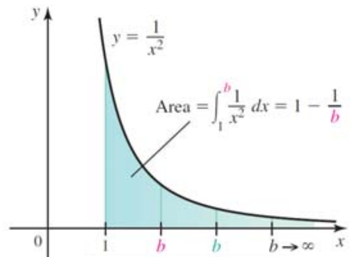
Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a real number such that $|f^{(4)}(x)| < K$ on $[a, b]$. The error in approximating the integral $\int_a^b f(x)\,dx$ by Simpson's Rule with n subintervals satisfies the inequality

$$E_S \le \frac{K(b-a)}{180}(\Delta x)^4.$$

8.8

Improper Integrals

Figure 8.17



What happens to the area under the curve as $b \to \infty$?

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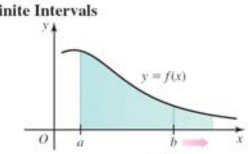
DEFINITIONS Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

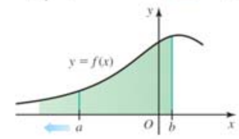
provided the limit exists.



2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

provided the limit exists.



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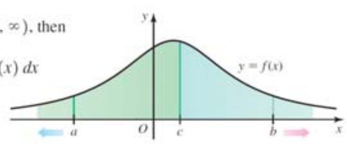
3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx,$$

provided both limits exist, where c is any real number.



In each case, if the limit exists, the improper integral is said to **converge**; if it does not exist, the improper integral is said to **diverge**.

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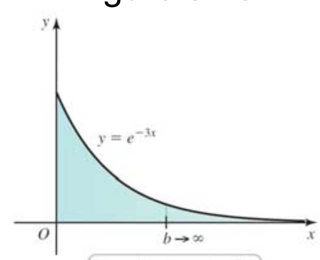
Figure 8.18



Area of region under the curve $y = e^{-3x}$ on $[0, \infty)$ has finite value $\frac{1}{3}$.

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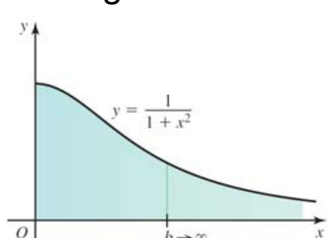
Figure 8.19



Area of region under the curve $y = \frac{1}{1+x^2}$ on $[0, \infty)$ has finite value $\frac{\pi}{2}$.

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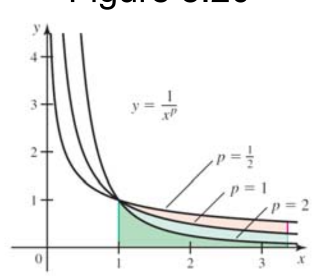
Figure 8.20



$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$, if $p > 1$.

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Figure 8.21

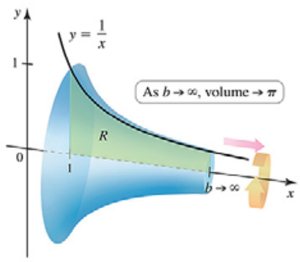


Figure 8.22

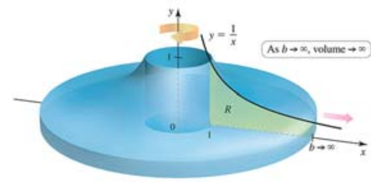
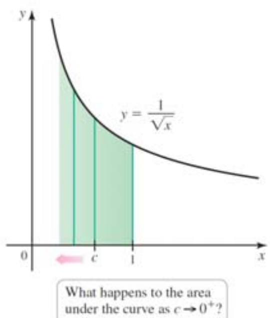


Figure 8.23

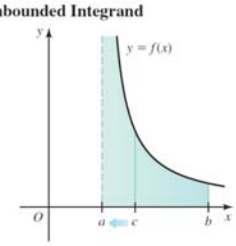


DEFINITIONS Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx,$$

provided the limit exists.



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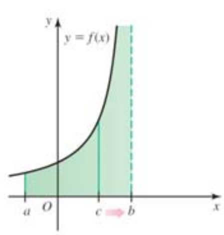
2. Suppose f is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx,$$

provided the limit exists.



continued…

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx,$$

provided the improper integrals on the right side exist.

In each case, if the limit exists, the improper integral is said to **converge**; if it does not exist, the improper integral is said to **diverge**.

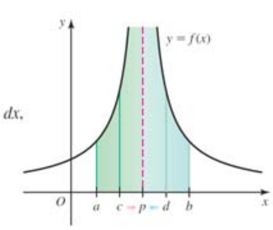


Figure 8.24

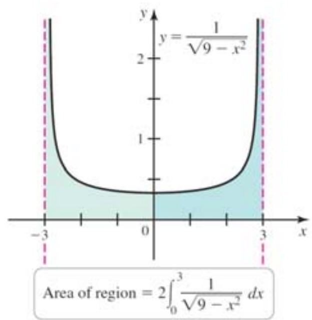


Figure 8.25

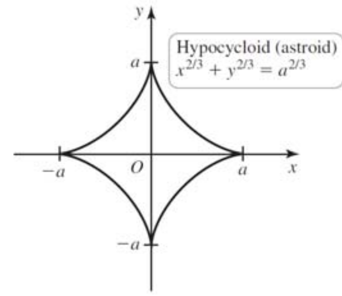


Figure 8.26