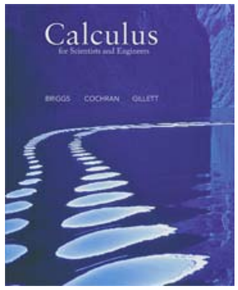
Chapter 9

Differential Equations



9.1

Basic Ideas

1

Figure 9.1

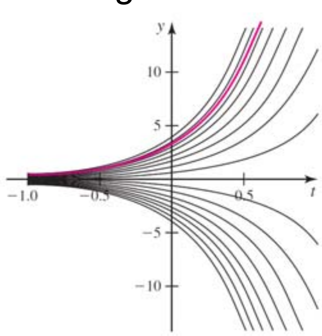


Figure 9.2

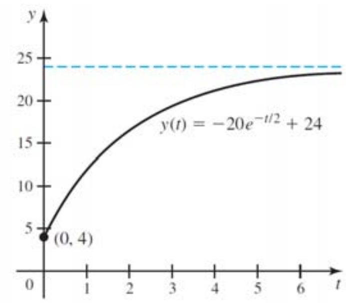
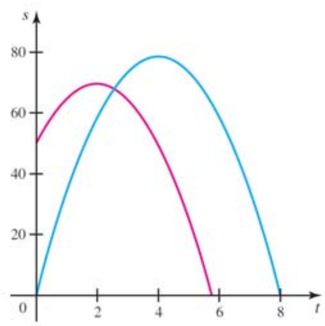


Figure 9.3



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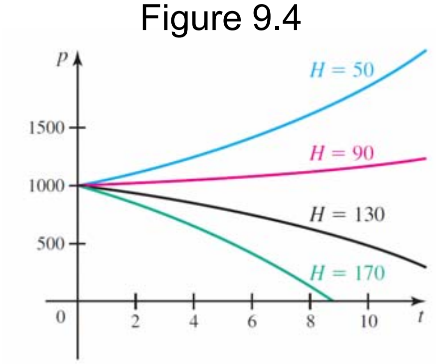

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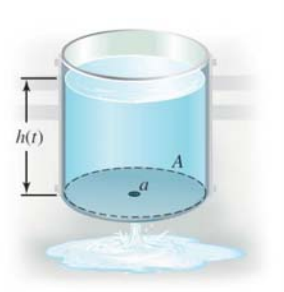

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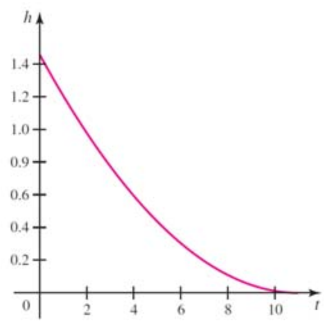
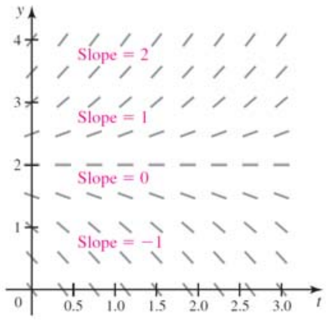


9.2

Direction Fields and Euler's Method

Figure 9.7



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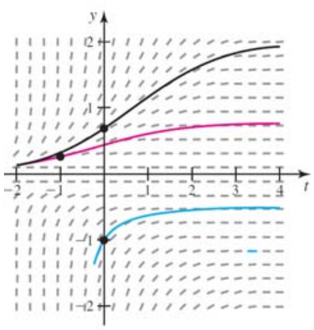


Figure 9.9

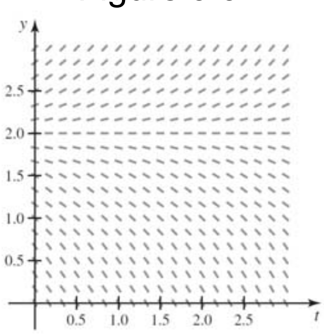
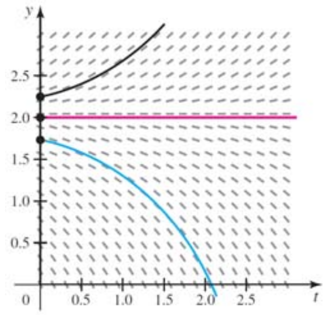


Figure 9.10



PROCEDURE **Sketching a Direction Field by Hand for $y'(t) = f(y)$**

1. Find the values of y for which $f(y) = 0$. For example, suppose that $f(a) = 0$. Then we have $y'(t) = 0$ whenever $y = a$, and the direction field at all points (t, a) consists of horizontal line segments. If the initial condition is $y(0) = a$, then the solution is $y(t) = a$, for all $t \geq 0$. Such a constant solution is called an **equilibrium solution**.
2. Find the values of y for which $f(y) > 0$. For example, suppose that $f(b) > 0$. Then $y'(t) > 0$ whenever $y = b$. It follows that the direction field at all points (t, b) has line segments with positive slopes, and the solution is increasing at those points.
3. Find the values of y for which $f(y) < 0$. For example, suppose that $f(c) < 0$. Then $y'(t) < 0$ whenever $y = c$. It follows that the direction field at all points (t, c) has line segments with negative slopes and the solution is decreasing at those points.

Figure 9.11

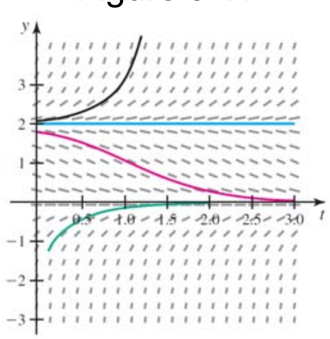


Figure 9.12

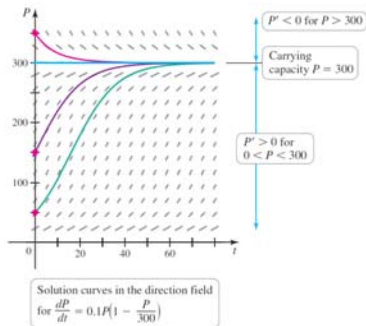


Figure 9.13

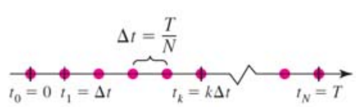
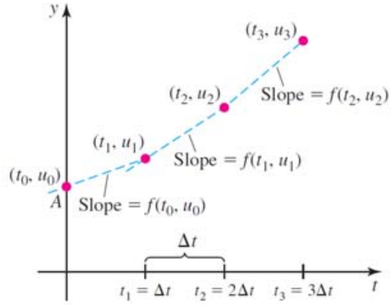


Figure 9.14



PROCEDURE **Euler's Method for $y'(t) = f(t, y), y(0) = A$ on $[0, T]$**

1. Choose either a time step Δt or a positive integer N such that $\Delta t = \frac{T}{N}$ and $t_k = k\Delta t$, for $k = 0, 1, 2, \ldots, N-1$.
2. Let $u_0 = y(0) = A$.
3. For $k = 0, 1, 2, \ldots, N-1$, compute

$$u_{k+1} = u_k + f(t_k, u_k)\Delta t.$$

Each u_k is an approximation to the exact solution $y(t_k)$.

Figure 9.15

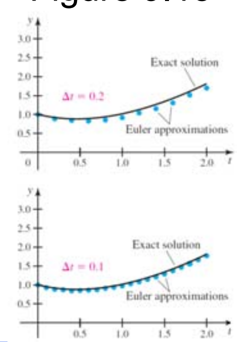


Table 9.1

t_k	$u_k(\Delta t = 0.2)$	$u_k(\Delta t = 0.1)$	$e_k(\Delta t = 0.2)$	$e_k(\Delta t = 0.1)$
0.0	1.000	1.000	0.000	0.000
0.2	0.900	0.913	0.0242	0.0117
0.4	0.850	0.873	0.0437	0.0211
0.6	0.845	0.875	0.0591	0.0286
0.8	0.881	0.917	0.0711	0.0345
1.0	0.952	0.994	0.0802	0.0390
1.2	1.057	1.102	0.0869	0.0423
1.4	1.191	1.238	0.0914	0.0446
1.6	1.352	1.401	0.0943	0.0460
1.8	1.537	1.586	0.0957	0.0468
2.0	1.743	1.792	0.0960	0.0470

9.3

Separable Differential Equations

Figure 9.16

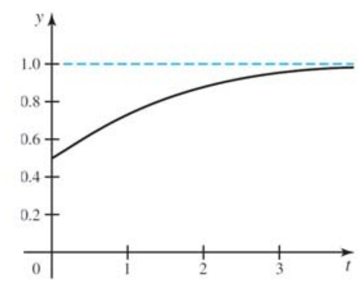


Figure 9.17

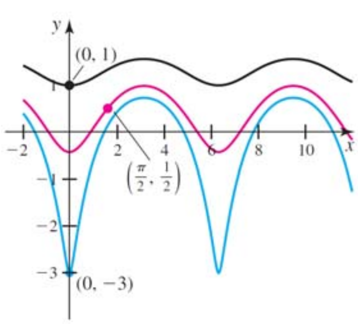


Figure 9.18

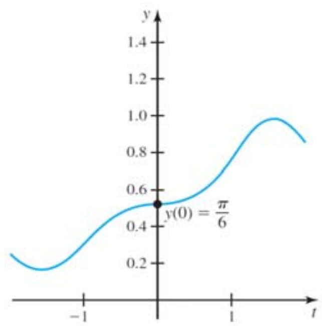
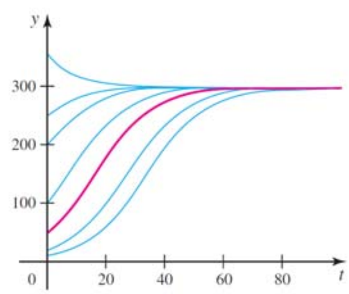


Figure 9.19



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9.4

Special First-Order Linear Differential Equations

SUMMARY Solution of a First-Order Linear Differential Equation

The general solution of the first-order linear equation $y'(t) = ky + b$, where $k \neq 0$ and b are real numbers, is

$$y(t) = Ce^{kt} - \frac{b}{k},$$

where C is an arbitrary constant. Given an initial condition, the value of C may be determined.

Figure 9.20

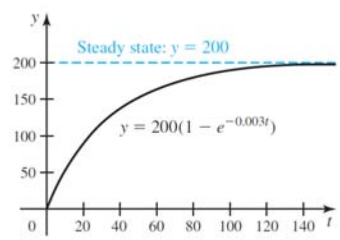


Figure 9.21

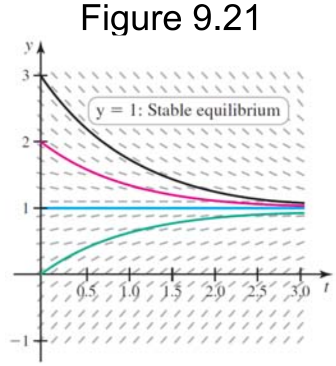
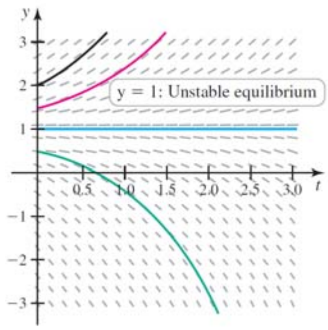


Figure 9.22



SUMMARY Equilibrium Solutions

The differential equation $y'(t) = f(y)$ has a (constant) **equilibrium** solution $y = a$ when $f(a) = 0$. The equilibrium is **stable** if initial conditions near $y = a$ produce solutions that approach $y = a$ as $t \to \infty$. The equilibrium is **unstable** if initial conditions near $y = a$ produce solutions that do not approach $y = a$ as $t \to \infty$.

Figure 9.23

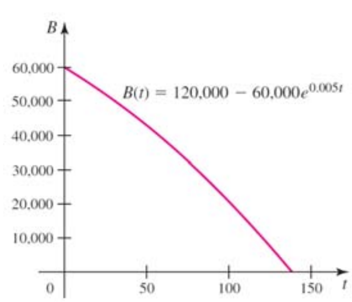
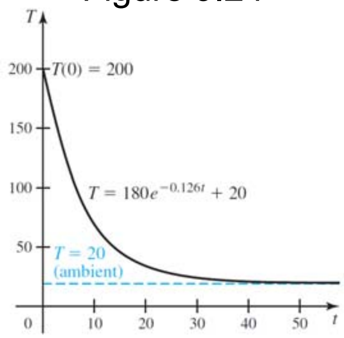


Figure 9.24



9.5

Modeling with
Differential Equations

Figure 9.25

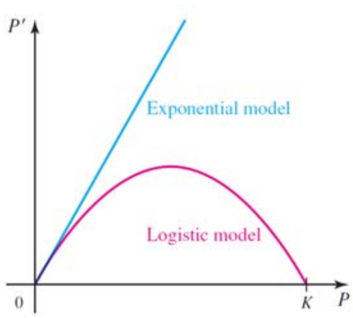
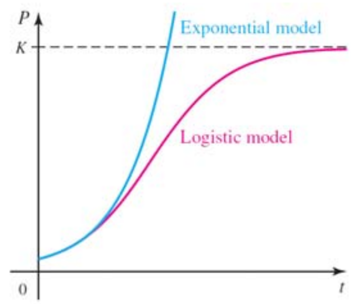


Figure 9.26



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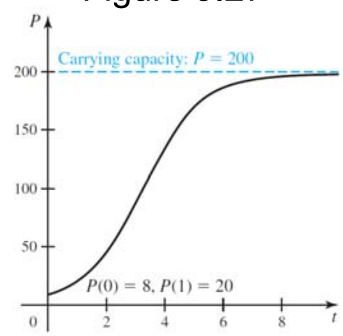

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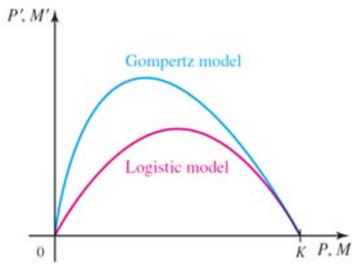

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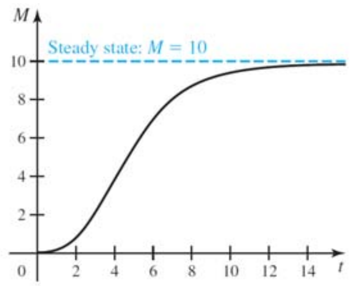

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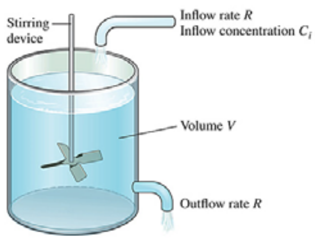

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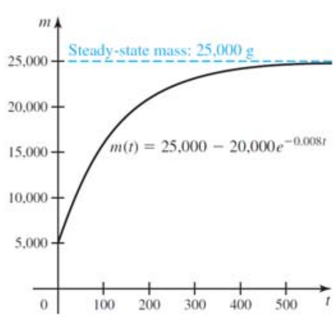
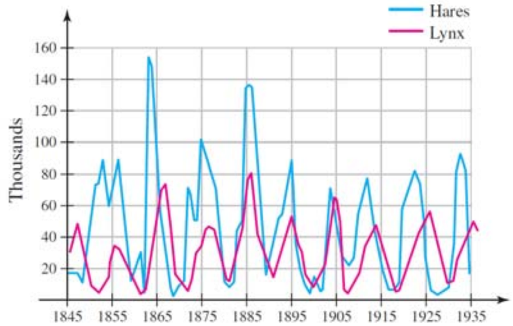


Figure 9.32



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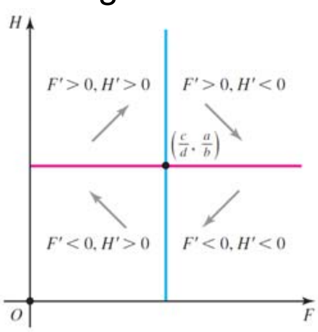

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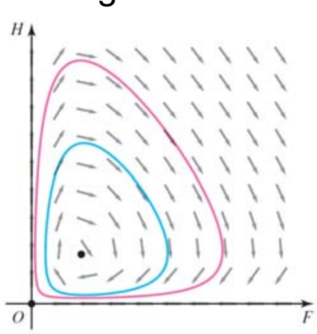

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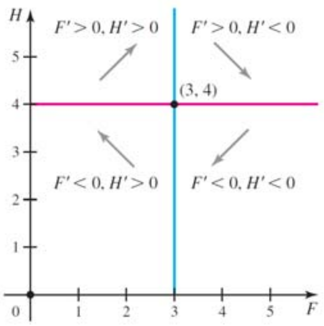

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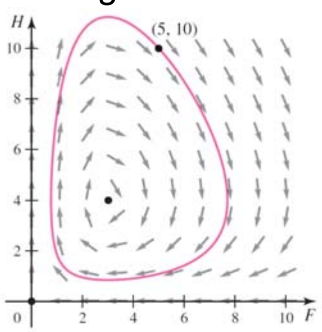

Figure 9.37