Chapter 9

Calculus

Differential Equations

9.1

Basic Ideas

Figure 9.1

Figure 9.2

$y(t) = -20e^{-t/2} + 24$

$(0, 4)$

Figure 9.3

Figure 9.4

$H = 50$
$H = 90$
$H = 130$
$H = 170$

1

Figure 9.5

Figure 9.6

9.2

Direction Fields and
Euler's Method

2

Figure 9.7
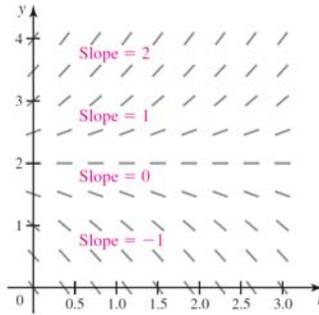
Figure 9.8

Figure 9.9

Figure 9.10

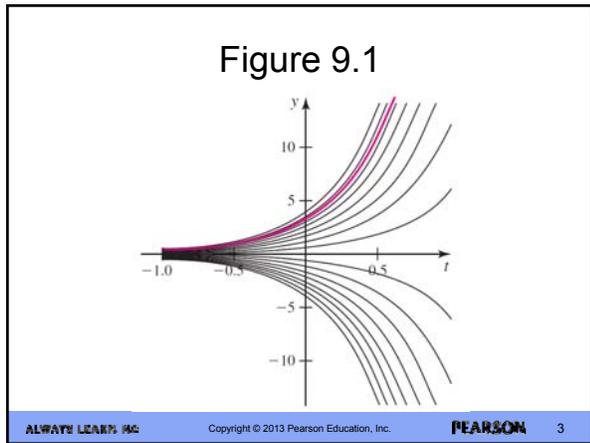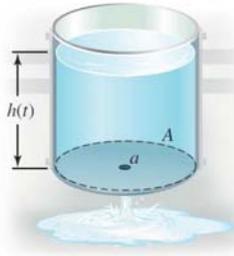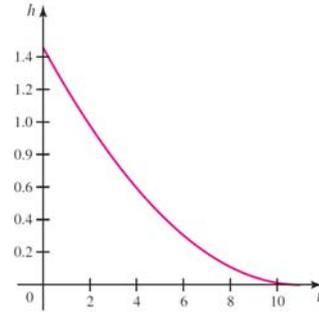**PROCEDURE** Sketching a Direction Field by Hand for $y'(t) = f(y)$

1. Find the values of y for which $f(y) = 0$. For example, suppose that $f(a) = 0$. Then we have $y'(t) = 0$ whenever $y = a$, and the direction field at all points (t, a) consists of horizontal line segments. If the initial condition is $y(0) = a$, then the solution is $y(t) = a$, for all $t \geq 0$. Such a constant solution is called an **equilibrium solution**.
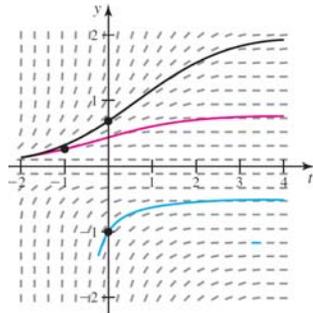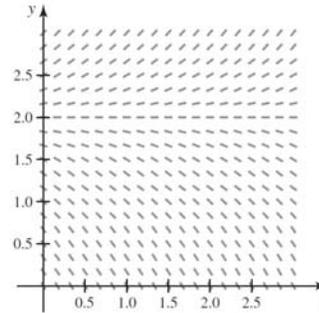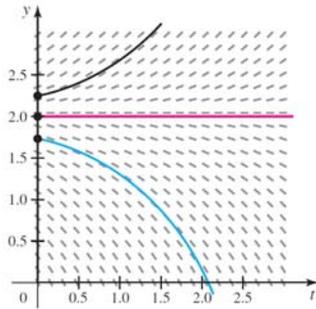
2. Find the values of y for which $f(y) > 0$. For example, suppose that $f(b) > 0$. Then $y'(t) > 0$ whenever $y = b$. It follows that the direction field at all points (t, b) has line segments with positive slopes, and the solution is increasing at those points.

3. Find the values of y for which $f(y) < 0$. For example, suppose that $f(c) < 0$. Then $y'(t) < 0$ whenever $y = c$. It follows that the direction field at all points (t, c) has line segments with negative slopes and the solution is decreasing at those points.
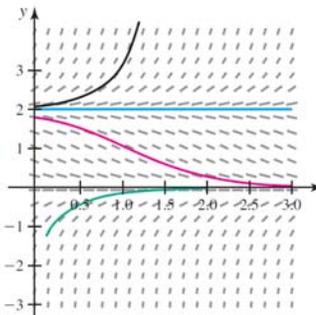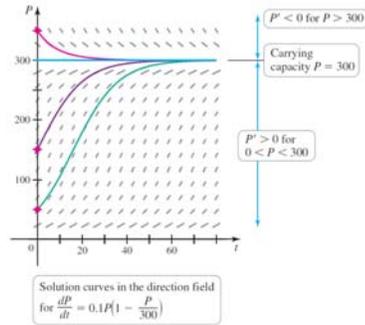
Figure 9.11

3

Figure 9.12

$P' < 0$ for $P > 300$

Carrying capacity $P = 300$

$P' > 0$ for $0 < P < 300$

Solution curves in the direction field for $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{300}\right)$

Figure 9.13

$\Delta t = \dfrac{T}{N}$

$t_0 = 0 \quad t_1 = \Delta t \quad\quad t_k = k\Delta t \quad\quad t_N = T$

Figure 9.14

(t_3, u_3)

(t_2, u_2) Slope $= f(t_2, u_2)$

(t_1, u_1)

(t_0, u_0) Slope $= f(t_1, u_1)$

A Slope $= f(t_0, u_0)$

Δt

$t_1 = \Delta t \quad t_2 = 2\Delta t \quad t_3 = 3\Delta t$
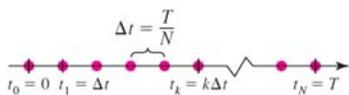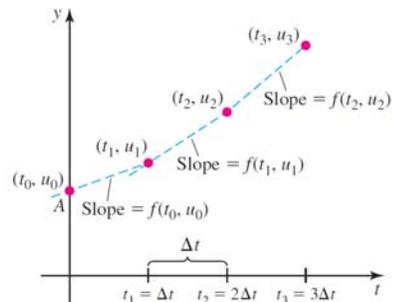
PROCEDURE Euler's Method for $y'(t) = f(t, y), y(0) = A$ on $[0, T]$

1. Choose either a time step Δt or a positive integer N such that $\Delta t = \dfrac{T}{N}$ and $t_k = k\Delta t$, for $k = 0, 1, 2, \ldots, N - 1$.
2. Let $u_0 = y(0) = A$.
3. For $k = 0, 1, 2, \ldots, N - 1$, compute
$$u_{k+1} = u_k + f(t_k, u_k)\Delta t.$$
Each u_k is an approximation to the exact solution $y(t_k)$.
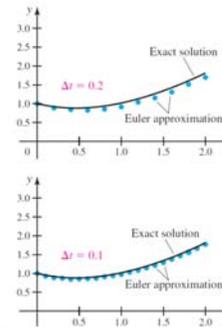
Figure 9.15

Table 9.1

t_k	$u_k (\Delta t = 0.2)$	$u_k (\Delta t = 0.1)$	$e_k (\Delta t = 0.2)$	$e_k (\Delta t = 0.1)$
0.0	1.000	1.000	0.000	0.000
0.2	0.900	0.913	0.0242	0.0117
0.4	0.850	0.873	0.0437	0.0211
0.6	0.845	0.875	0.0591	0.0286
0.8	0.881	0.917	0.0711	0.0345
1.0	0.952	0.994	0.0802	0.0390
1.2	1.057	1.102	0.0869	0.0423
1.4	1.191	1.238	0.0914	0.0446
1.6	1.352	1.401	0.0943	0.0460
1.8	1.537	1.586	0.0957	0.0468
2.0	1.743	1.792	0.0960	0.0470

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9.3

Separable Differential Equations

Figure 9.16
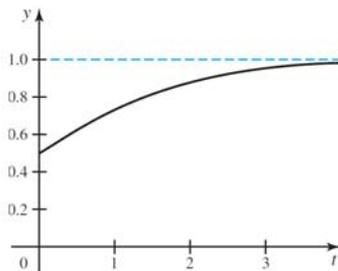
Figure 9.17

Figure 9.18

Figure 9.18 graph showing curve with point $y(0) = \dfrac{\pi}{6}$

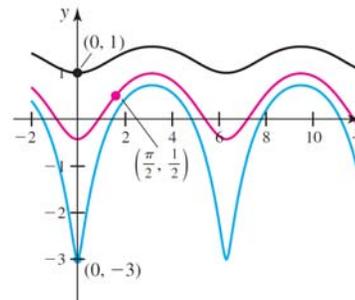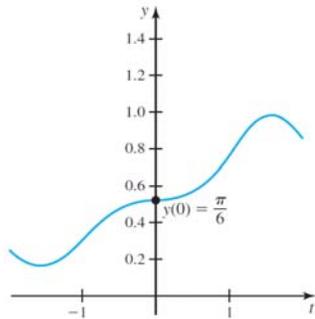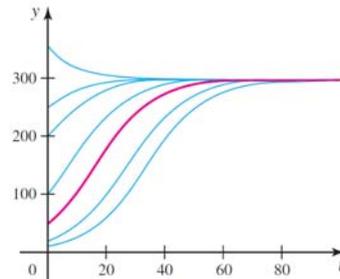## Figure 9.19

9.4

Special First-Order Linear Differential Equations

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SUMMARY   Solution of a First-Order Linear Differential Equation

The general solution of the first-order linear equation $y'(t) = ky + b$, where $k \neq 0$ and b are real numbers, is

$$y(t) = Ce^{kt} - \frac{b}{k},$$

where C is an arbitrary constant. Given an initial condition, the value of C may be determined.
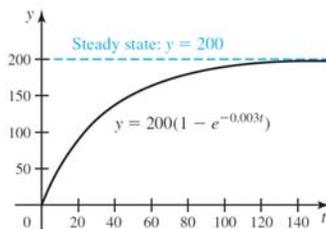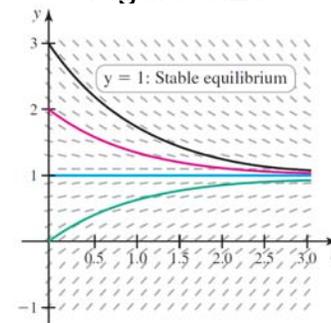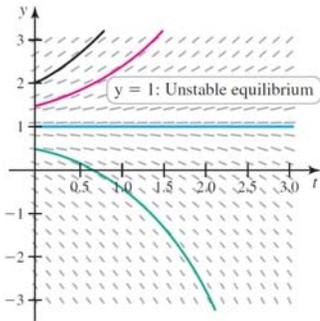
Figure 9.20

Steady state: $y = 200$

$y = 200(1 - e^{-0.003t})$

Figure 9.21

$y = 1$: Stable equilibrium

Figure 9.22

y = 1: Unstable equilibrium

SUMMARY   Equilibrium Solutions

The differential equation $y'(t) = f(y)$ has a (constant) **equilibrium** solution $y = a$ when $f(a) = 0$. The equilibrium is **stable** if initial conditions near $y = a$ produce solutions that approach $y = a$ as $t \to \infty$. The equilibrium is **unstable** if initial conditions near $y = a$ produce solutions that do not approach $y = a$ as $t \to \infty$.
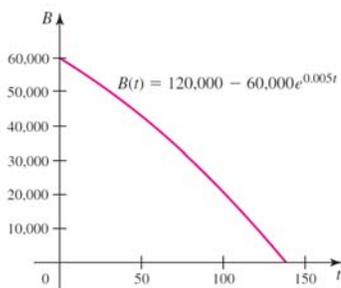
Figure 9.23

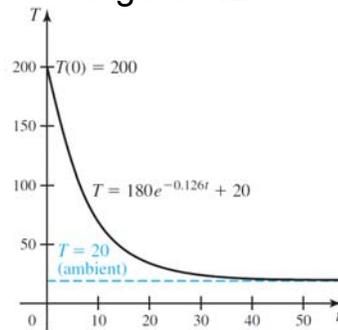$B(t) = 120{,}000 - 60{,}000e^{0.005t}$

6

Figure 9.24

$T(0) = 200$

$T = 180e^{-0.126t} + 20$

$T = 20$ (ambient)

9.5

Modeling with Differential Equations

Figure 9.25
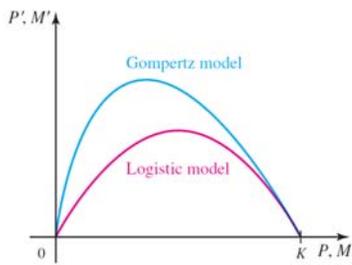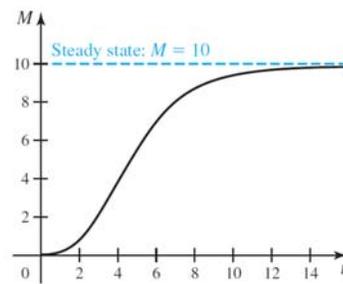

Exponential model

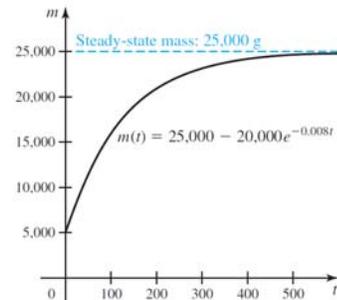Logistic model
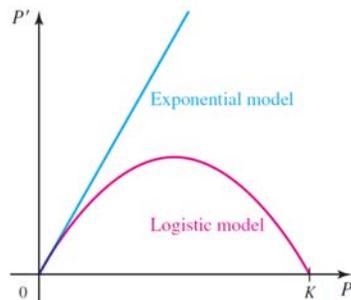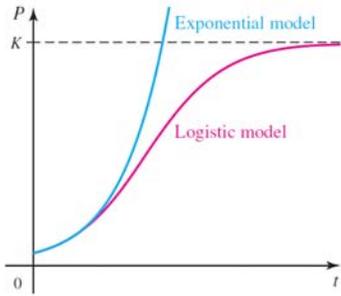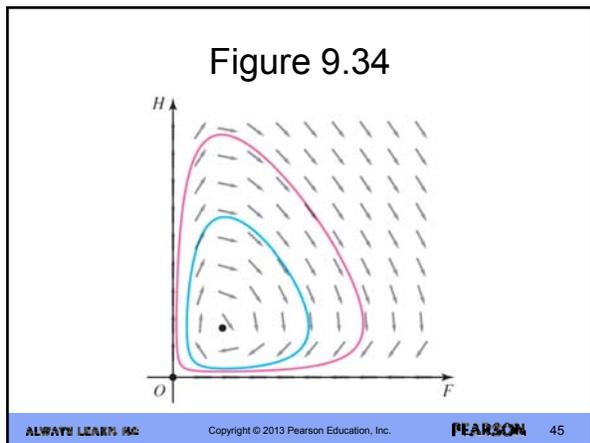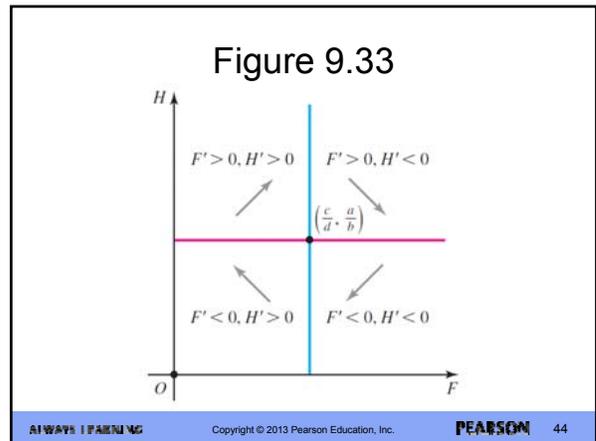
Figure 9.26

Figure 9.27

Figure 9.28

7

Figure 9.29

Figure 9.30

Figure 9.31

Figure 9.32

Figure 9.33

$F' > 0, H' > 0$ $F' > 0, H' < 0$

$\left(\dfrac{c}{d}, \dfrac{a}{b}\right)$

$F' < 0, H' > 0$ $F' < 0, H' < 0$

Figure 9.34

8

Figure 9.35

$F' > 0, H' > 0$ $F' > 0, H' < 0$

$(3, 4)$

$F' < 0, H' > 0$ $F' < 0, H' < 0$

Figure 9.36

$(5, 10)$

Figure 9.37