Exercise Set 1.8

► In Exercises 1–2, find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$. ◄

1. (a) A has size 3×2. (b) A has size 2×3.
(c) A has size 3×3. (d) A has size 1×6.

2. (a) A has size 4×5. (b) A has size 5×4.
(c) A has size 4×4. (d) A has size 3×1.

► In Exercises 3–4, find the domain and codomain of the transformation defined by the equations. ◄

3. (a) $\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$ (b) $\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$

4. (a) $\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$ (b) $\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$

► In Exercises 5–6, find the domain and codomain of the transformation defined by the matrix product. ◄

5. (a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

6. (a) $\begin{bmatrix} 6 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & -6 \\ 3 & 7 & -4 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

► In Exercises 7–8, find the domain and codomain of the transformation T defined by the formula. ◄

7. (a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
(b) $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$

8. (a) $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$
(b) $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

► In Exercises 9–10, find the domain and codomain of the transformation T defined by the formula. ◄

9. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 \\ x_1 - x_2 \\ 3x_2 \end{bmatrix}$ **10.** $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 - x_3 \\ 0 \end{bmatrix}$

► In Exercises 11–12, find the standard matrix for the transformation defined by the equations. ◄

11. (a) $\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$ (b) $\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= - x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$

12. (a) $\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$ (b) $\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$

13. Find the standard matrix for the transformation T defined by the formula.

(a) $T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2)$
(b) $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$
(c) $T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$
(d) $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$

14. Find the standard matrix for the operator T defined by the formula.

(a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
(b) $T(x_1, x_2) = (x_1, x_2)$
(c) $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$
(d) $T(x_1, x_2, x_3) = (4x_1, 7x_2, -8x_3)$

15. Find the standard matrix for the operator $T: R^3 \to R^3$ defined by

$$\begin{aligned} w_1 &= 3x_1 + 5x_2 - x_3 \\ w_2 &= 4x_1 - x_2 + x_3 \\ w_3 &= 3x_1 + 2x_2 - x_3 \end{aligned}$$

and then compute $T(-1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

16. Find the standard matrix for the transformation $T: R^4 \to R^2$ defined by

$$\begin{aligned} w_1 &= 2x_1 + 3x_2 - 5x_3 - x_4 \\ w_2 &= x_1 - 5x_2 + 2x_3 - 3x_4 \end{aligned}$$

and then compute $T(1, -1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

► In Exercises 17–18, find the standard matrix for the transformation and use it to compute $T(\mathbf{x})$. Check your result by substituting directly in the formula for T. ◄

17. (a) $T(x_1, x_2) = (-x_1 + x_2, x_2)$; $\mathbf{x} = (-1, 4)$
(b) $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + x_3, 0)$; $\mathbf{x} = (2, 1, -3)$

18. (a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$; $\mathbf{x} = (-2, 2)$
(b) $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$; $\mathbf{x} = (1, 0, 5)$

► In Exercises 19–20, find $T_A(\mathbf{x})$, and express your answer in matrix form. ◄

19. (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

20. (a) $A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

▶ In Exercises 21–22, use Theorem 1.8.2 to show that T is a matrix transformation. ◀

21. (a) $T(x, y) = (2x + y, x - y)$

(b) $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

22. (a) $T(x, y, z) = (x + y, y + z, x)$

(b) $T(x_1, x_2) = (x_2, x_1)$

▶ In Exercises 23–24, use Theorem 1.8.2 to show that T is not a matrix transformation. ◀

23. (a) $T(x, y) = (x^2, y)$

(b) $T(x, y, z) = (x, y, xz)$

24. (a) $T(x, y) = (x, y + 1)$

(b) $T(x_1, x_2, x_3) = \left(x_1, x_2, \sqrt{x_3}\right)$

25. A function of the form $f(x) = mx + b$ is commonly called a "linear function" because the graph of $y = mx + b$ is a line. Is f a matrix transformation on R?

26. Show that $T(x, y) = (0, 0)$ defines a matrix operator on R^2 but $T(x, y) = (1, 1)$ does not.

▶ In Exercises 27–28, the images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$. ◀

27. $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

28. $T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

29. Let $T: R^2 \to R^2$ be a linear operator for which the images of the standard basis vectors for R^2 are $T(\mathbf{e}_1) = (a, b)$ and $T(\mathbf{e}_2) = (c, d)$. Find $T(1, 1)$.

30. We proved in the text that if $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$. Show that the converse of this result is false by finding a mapping $T: R^n \to R^m$ that is not a matrix transformation but for which $T(\mathbf{0}) = \mathbf{0}$.

31. Let $T_A: R^3 \to R^3$ be multiplication by

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}$$

and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for R^3. Find the following vectors by inspection.

(a) $T_A(\mathbf{e}_1)$, $T_A(\mathbf{e}_2)$, and $T_A(\mathbf{e}_3)$

(b) $T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ (c) $T_A(7\mathbf{e}_3)$

Working with Proofs

32. (a) Prove: If $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$; that is, T maps the zero vector in R^n into the zero vector in R^m.

(b) The converse of this is not true. Find an example of a function T for which $T(\mathbf{0}) = \mathbf{0}$ but which is not a matrix transformation.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If A is a 2×3 matrix, then the domain of the transformation T_A is R^2.

(b) If A is an $m \times n$ matrix, then the codomain of the transformation T_A is R^n.

(c) There is at least one linear transformation $T: R^n \to R^m$ for which $T(2\mathbf{x}) = 4T(\mathbf{x})$ for some vector $\mathbf{x}$ in R^n.

(d) There are linear transformations from R^n to R^m that are not matrix transformations.

(e) If $T_A: R^n \to R^n$ and if $T_A(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^n, then A is the $n \times n$ zero matrix.

(f) There is only one matrix transformation $T: R^n \to R^m$ such that $T(-\mathbf{x}) = -T(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n.

(g) If $\mathbf{b}$ is a nonzero vector in R^n, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a matrix operator on R^n.