More on the Equivalence Theorem

As our final result in this section, we will add parts (*b*), (*c*), and (*d*) of Theorem 4.10.1 to Theorem 4.8.8.

THEOREM 4.10.2 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are linearly independent.*

(*i*) *The row vectors of A are linearly independent.*

(*j*) *The column vectors of A span R^n.*

(*k*) *The row vectors of A span R^n.*

(*l*) *The column vectors of A form a basis for R^n.*

(*m*) *The row vectors of A form a basis for R^n.*

(*n*) *A has rank n.*

(*o*) *A has nullity* 0.

(*p*) *The orthogonal complement of the null space of A is R^n.*

(*q*) *The orthogonal complement of the row space of A is* $\{\mathbf{0}\}$.

(*r*) *The kernel of T_A is* $\{\mathbf{0}\}$.

(*s*) *The range of T_A is R^n.*

(*t*) *T_A is one-to-one.*

Exercise Set 4.10

▶ In Exercises 1–4, determine whether the operators T_1 and T_2 commute; that is, whether $T_1 \circ T_2 = T_2 \circ T_1$. ◀

1. (a) $T_1: R^2 \to R^2$ is the reflection about the line $y = x$, and $T_2: R^2 \to R^2$ is the orthogonal projection onto the x-axis.

(b) $T_1: R^2 \to R^2$ is the reflection about the x-axis, and $T_2: R^2 \to R^2$ is the reflection about the line $y = x$.

2. (a) $T_1: R^2 \to R^2$ is the orthogonal projection onto the x-axis, and $T_2: R^2 \to R^2$ is the orthogonal projection onto the y-axis.

(b) $T_1: R^2 \to R^2$ is the rotation about the origin through an angle of $\pi/4$, and $T_2: R^2 \to R^2$ is the reflection about the y-axis.

3. $T_1: R^3 \to R^3$ is a dilation with factor k, and $T_2: R^3 \to R^3$ is a contraction with factor $1/k$.

4. $T_1: R^3 \to R^3$ is the rotation about the x-axis through an angle θ_1, and $T_2: R^3 \to R^3$ is the rotation about the z-axis through an angle θ_2.

▶ In Exercises 5–6, let T_A and T_B bet the operators whose standard matrices are given. Find the standard matrices for $T_B \circ T_A$ and $T_A \circ T_B$. ◀

5. $A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 4 \\ -1 & 5 & 2 \\ 2 & -3 & 8 \end{bmatrix}$

7. Find the standard matrix for the stated composition in R^2.

(a) A rotation of 90°, followed by a reflection about the line $y = x$.

(b) An orthogonal projection onto the y-axis, followed by a contraction with factor $k = \frac{1}{2}$.

(c) A reflection about the x-axis, followed by a dilation with factor $k = 3$, followed by a rotation about the origin of 60°.

8. Find the standard matrix for the stated composition in R^2.

(a) A rotation about the origin of 60°, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.

(b) A dilation with factor $k = 2$, followed by a rotation about the origin of 45°, followed by a reflection about the y-axis.

(c) A rotation about the origin of 15°, followed by a rotation about the origin of 105°, followed by a rotation about the origin of 60°.

9. Find the standard matrix for the stated composition in R^3.

(a) A reflection about the yz-plane, followed by an orthogonal projection onto the xz-plane.

(b) A rotation of 45° about the y-axis, followed by a dilation with factor $k = \sqrt{2}$.

(c) An orthogonal projection onto the xy-plane, followed by a reflection about the yz-plane.

10. Find the standard matrix for the stated composition in R^3.

(a) A rotation of 30° about the x-axis, followed by a rotation of 30° about the z-axis, followed by a contraction with factor $k = \frac{1}{4}$.

(b) A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection onto the yz-plane.

(c) A rotation of 270° about the x-axis, followed by a rotation of 90° about the y-axis, followed by a rotation of 180° about the z-axis.

11. Let $T_1(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ and $T_2(x_1, x_2) = (3x_1, 2x_1 + 4x_2)$.

(a) Find the standard matrices for T_1 and T_2.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2))$.

12. Let $T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$.

(a) Find the standard matrices for T_1 and T_2.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

▶ In Exercises 13–14, determine by inspection whether the stated matrix operator is one-to-one. ◀

13. (a) The orthogonal projection onto the x-axis in R^2.

(b) The reflection about the y-axis in R^2.

(c) The reflection about the line $y = x$ in R^2.

(d) A contraction with factor $k > 0$ in R^2.

14. (a) A rotation about the z-axis in R^3.

(b) A reflection about the xy-plane in R^3.

(c) A dilation with factor $k > 0$ in R^3.

(d) An orthogonal projection onto the xz-plane in R^3.

▶ In Exercises 15–16, describe in words the inverse of the given one-to-one operator. ◀

15. (a) The reflection about the x-axis on R^2.

(b) The rotation about the origin through an angle of $\pi/4$ on R^2.

(c) The dilation with factor of 3 on R^2.

16. (a) The reflection about the yz-plane in R^3.

(b) The contraction with factor $\frac{1}{5}$ in R^3.

(c) The rotation through an angle of $-18°$ about the z-axis in R^3.

▶ In Exercises 17–18, express the equations in matrix form, and then use parts (g) and (s) of Theorem 4.10.2 to determine whether the operator defined by the equations is one-to-one. ◀

17. (a) $\begin{aligned} w_1 &= 8x_1 + 4x_2 \\ w_2 &= 2x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= -x_1 + 3x_2 + 2x_3 \\ w_2 &= 2x_1 + 4x_3 \\ w_3 &= x_1 + 3x_2 + 6x_3 \end{aligned}$

18. (a) $\begin{aligned} w_1 &= 2x_1 - 3x_2 \\ w_2 &= 5x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= x_1 + 2x_2 + 3x_3 \\ w_2 &= 2x_1 + 5x_2 + 3x_3 \\ w_3 &= x_1 + 8x_3 \end{aligned}$

19. Determine whether the matrix operator $T: R^2 \to R^2$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2)$.

(a) $\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$

20. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

(a) $\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$

▶ In Exercises 21–22, determine whether multiplication by A is a one-to-one matrix transformation. ◀

21. (a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{bmatrix}$

22. (a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$

▶ In Exercises 23–24, let T be multiplication by the matrix A. Find

(a) a basis for the range of T.

(b) a basis for the kernel of T.

(c) the rank and nullity of T.

(d) the rank and nullity of A. ◀

23. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{bmatrix}$

▶ In Exercises 25–26, let $T_A: R^4 \to R^3$ be multiplication by A. Find a basis for the kernel of T_A, and then find a basis for the range of T_A that consists of column vectors of A. ◀

25. $A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{bmatrix}$

27. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T: R^n \to R^n$ be multiplication by A.

(a) What can you say about the range of the matrix operator T? Give an example that illustrates your conclusion.

(b) What can you say about the number of vectors that T maps into $\mathbf{0}$?

28. Answer the questions in Exercise 27 in the case where $\det(A) \neq 0$.

29. (a) Is a composition of one-to-one matrix transformations one-to-one? Justify your conclusion.

(b) Can the composition of a one-to-one matrix transformation and a matrix transformation that is not one-to-one be one-to-one? Account for both possible orders of composition and justify your conclusion.

30. Let $T_A: R^2 \to R^2$ be multiplication by

$$A = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

(a) What is the geometric effect of applying this transformation to a vector $\mathbf{x}$ in R^2?

(b) Express the operator T_A as a composition of two linear operators on R^2.

▶ In Exercises 31–32, use matrix inversion to confirm the stated result in R^2. ◀

31. (a) The inverse transformation for a reflection about $y = x$ is a reflection about $y = x$.

(b) The inverse transformation for a compression along an axis is an expansion along that axis.

32. (a) The inverse transformation for a reflections about a coordinate axis is a reflection about that axis.

(b) The inverse transformation for a shear along a coordinate axis is a shear along that axis.

Working with Proofs

33. Prove that the matrix transformations T_A and T_B commute if and only if the matrices A and B commute.

34. Prove the implication $(c) \Rightarrow (d)$ in Theorem 4.10.1.

35. Prove the implication $(d) \Rightarrow (a)$ in Theorem 4.10.1.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If T_A and T_B are matrix operators on R^n, then $T_A(T_B(\mathbf{x})) = T_B(T_A(\mathbf{x}))$ for every vector $\mathbf{x}$ in R^n.

(b) If T_1 and T_2 are matrix operators on R^n, then $[T_2 \circ T_1] = [T_2][T_1]$.

(c) A composition of two rotation operators about the origin of R^2 is another rotation about the origin.

(d) A composition of two reflection operators in R^2 is another reflection operator.

(e) The kernel of a matrix transformation $T_A: R^n \to R^m$ is the same as the null space of A.

(f) If there is a nonzero vector in the kernel of the matrix operator $T_A: R^n \to R^n$, then this operator is not one-to-one.

(g) If A is an $n \times n$ matrix and if the linear system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then the range of the matrix operator is not R^n.

Working with Technology

T1. (a) Find the standard matrix for the linear operator on R^3 that performs a counterclockwise rotation of 47° about the x-axis, followed by a counterclockwise rotation of 68° about the y-axis, followed by a counterclockwise rotation of 33° about the z-axis.

(b) Find the image of the point (1, 1, 1) under the operator in part (a).

T2. Find the standard matrix for the linear operator on R^2 that first reflects each point in the plane about the line through the origin that makes an angle of 27° with the positive x-axis and then projects the resulting point orthogonally onto the line through the origin that makes an angle of 51° with the positive x-axis.