Exercise Set 6.3

1. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^2.

(a) $(0, 1),\ (2, 0)$

(b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(d) $(0, 0),\ (0, 1)$

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right),\ \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right),\ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

(c) $(1, 0, 0),\ \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ (0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

3. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on P_2 (see Example 7 of Section 6.1).

(a) $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2,\ \ p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2,$
$p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

(b) $p_1(x) = 1,\ \ p_2(x) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ \ p_3(x) = x^2$

4. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on M_{22} (see Example 6 of Section 6.1).

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix},\ \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix},\ \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

▶ In Exercises 5–6, show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space. ◀

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & 0 & -\frac{2}{3} \end{bmatrix}$

7. Verify that the vectors

$$\mathbf{v}_1 = \left(-\tfrac{3}{5}, \tfrac{4}{5}, 0\right),\ \ \mathbf{v}_2 = \left(\tfrac{4}{5}, \tfrac{3}{5}, 0\right),\ \ \mathbf{v}_3 = (0, 0, 1)$$

form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

8. Use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ in Exercise 7.

9. Verify that the vectors

$$\mathbf{v}_1 = (2, -2, 1),\quad \mathbf{v}_2 = (2, 1, -2),\quad \mathbf{v}_3 = (1, 2, 2)$$

form an orthogonal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (-1, 0, 2)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

10. Verify that the vectors

$$\mathbf{v}_1 = (1, -1, 2, -1),\quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1),\quad \mathbf{v}_4 = (1, 0, 0, 1)$$

form an orthogonal basis for R^4 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (1, 1, 1, 1)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

▶ In Exercises 11–14, find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in the stated exercise. ◀

11. Exercise 7

12. Exercise 8

13. Exercise 9

14. Exercise 10

▶ In Exercises 15–18, let R^2 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the line spanned by the vector $\mathbf{v}$.

(b) Find the component of $\mathbf{u}$ orthogonal to the line spanned by the vector $\mathbf{v}$, and confirm that this component is orthogonal to the line. ◀

15. $\mathbf{u} = (-1, 6);\ \ \mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$

16. $\mathbf{u} = (2, 3);\ \ \mathbf{v} = \left(\frac{5}{13}, \frac{12}{13}\right)$

17. $\mathbf{u} = (2, 3);\ \ \mathbf{v} = (1, 1)$

18. $\mathbf{u} = (3, -1);\ \ \mathbf{v} = (3, 4)$

▶ In Exercises 19–22, let R^3 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Find the component of $\mathbf{u}$ orthogonal to the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and confirm that this component is orthogonal to the plane. ◀

19. $\mathbf{u} = (4, 2, 1);\ \ \mathbf{v}_1 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right),\ \ \mathbf{v}_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

20. $\mathbf{u} = (3, -1, 2);\ \ \mathbf{v}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right),\ \ \mathbf{v}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

21. $\mathbf{u} = (1, 0, 3);\ \ \mathbf{v}_1 = (1, -2, 1),\ \ \mathbf{v}_2 = (2, 1, 0)$

22. $\mathbf{u} = (1, 0, 2);\ \ \mathbf{v}_1 = (3, 1, 2),\ \ \mathbf{v}_2 = (-1, 1, 1)$

▶ In Exercises 23–24, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -2)$ on the subspace W spanned by these vectors. ◀

23. $\mathbf{v}_1 = (1, 1, 1, 1),\ \ \mathbf{v}_2 = (1, 1, -1, -1)$

24. $\mathbf{v}_1 = (0, 1, -4, -1),\ \ \mathbf{v}_2 = (3, 5, 1, 1)$

▶ In Exercises 25–26, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -1)$ onto the subspace W spanned by these vectors. ◀

25. $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

26. $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

▶ In Exercises 27–28, let R^2 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis. Draw both sets of basis vectors in the xy-plane. ◀

27. $\mathbf{u}_1 = (1, -3)$, $\mathbf{u}_2 = (2, 2)$ **28.** $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (3, -5)$

▶ In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis. ◀

29. $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

30. $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (3, 7, -2)$, $\mathbf{u}_3 = (0, 4, 1)$

31. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, 2, 1, 0), \quad \mathbf{u}_2 = (1, -1, 0, 0),$$
$$\mathbf{u}_3 = (1, 2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, 1)$$

32. Let R^3 have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by $(0, 1, 2)$, $(-1, 0, 1)$, $(-1, 1, 3)$.

33. Let $\mathbf{b}$ and W be as in Exercise 23. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

34. Let $\mathbf{b}$ and W be as in Exercise 25. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

35. Let R^3 have the Euclidean inner product. The subspace of R^3 spanned by the vectors $\mathbf{u}_1 = (1, 1, 1)$ and $\mathbf{u}_2 = (2, 0, -1)$ is a plane passing through the origin. Express $\mathbf{w} = (1, 2, 3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ lies in the plane and $\mathbf{w}_2$ is perpendicular to the plane.

36. Let R^4 have the Euclidean inner product. Express the vector $\mathbf{w} = (-1, 2, 6, 0)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in the space W spanned by $\mathbf{u}_1 = (-1, 0, 1, 2)$ and $\mathbf{u}_2 = (0, 1, 0, 1)$, and $\mathbf{w}_2$ is orthogonal to W.

37. Let R^3 have the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$$

Use the Gram–Schmidt process to transform $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

38. Verify that the set of vectors $\{(1, 0), (0, 1)\}$ is orthogonal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + u_2v_2$ on R^2; then convert it to an orthonormal set by normalizing the vectors.

39. Find vectors $\mathbf{x}$ and $\mathbf{y}$ in R^2 that are orthonormal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ but are not orthonormal with respect to the Euclidean inner product.

40. In Example 3 of Section 4.9 we found the orthogonal projection of the vector $\mathbf{x} = (1, 5)$ onto the line through the origin making an angle of $\pi/6$ radians with the positive x-axis. Solve that same problem using Theorem 6.3.4.

41. This exercise illustrates that the orthogonal projection resulting from Formula (12) in Theorem 6.3.4 does not depend on which orthogonal basis vectors are used.

(a) Let R^3 have the Euclidean inner product, and let W be the subspace of R^3 spanned by the orthogonal vectors

$$\mathbf{v}_1 = (1, 0, 1) \quad \text{and} \quad \mathbf{v}_2 = (0, 1, 0)$$

Show that the orthogonal vectors

$$\mathbf{v}'_1 = (1, 1, 1) \quad \text{and} \quad \mathbf{v}'_2 = (1, -2, 1)$$

span the same subspace W.

(b) Let $\mathbf{u} = (-3, 1, 7)$ and show that the same vector $\text{proj}_W \mathbf{u}$ results regardless of which of the bases in part (a) is used for its computation.

42. (***Calculus required***) Use Theorem 6.3.2(*a*) to express the following polynomials as linear combinations of the first three Legendre polynomials (see the Remark following Example 9).

(a) $1 + x + 4x^2$ (b) $2 - 7x^2$ (c) $4 + 3x$

43. (***Calculus required***) Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$$

Apply the Gram–Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis.

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

▶ In Exercises 45–48, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A. ◀

45. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

46. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$

49. Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

50. In the Remark following Example 8 we discussed two alternative ways to perform the calculations in the Gram–Schmidt process: normalizing each orthogonal basis vector as soon as it is calculated and scaling the orthogonal basis vectors at each step to eliminate fractions. Try these methods in Example 8.

Working with Proofs

51. Prove part (*a*) of Theorem 6.3.6.

52. In Step 3 of the proof of Theorem 6.3.5, it was stated that "the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$." Prove this statement.

53. Prove that the diagonal entries of R in Formula (16) are nonzero.

54. Show that matrix Q in Example 10 has the property $QQ^T = I_3$, and prove that every $m \times n$ matrix Q with orthonormal column vectors has the property $QQ^T = I_m$.

55. (a) Prove that if W is a subspace of a finite-dimensional vector space V, then the mapping $T: V \to W$ defined by $T(\mathbf{v}) = \text{proj}_W \mathbf{v}$ is a linear transformation.

(b) What are the range and kernel of the transformation in part (a)?

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) Every linearly independent set of vectors in an inner product space is orthogonal.

(b) Every orthogonal set of vectors in an inner product space is linearly independent.

(c) Every nontrivial subspace of R^3 has an orthonormal basis with respect to the Euclidean inner product.

(d) Every nonzero finite-dimensional inner product space has an orthonormal basis.

(e) $\text{proj}_W \mathbf{x}$ is orthogonal to every vector of W.

(f) If A is an $n \times n$ matrix with a nonzero determinant, then A has a QR-decomposition.

Working with Technology

T1. (a) Use the Gram–Schmidt process to find an orthonormal basis relative to the Euclidean inner product for the column space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

(b) Use the method of Example 9 to find a QR-decomposition of A.

T2. Let P_4 have the evaluation inner product at the points $-2, -1, 0, 1, 2$. Find an orthogonal basis for P_4 relative to this inner product by applying the Gram–Schmidt process to the vectors

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \quad \mathbf{p}_3 = x^3, \quad \mathbf{p}_4 = x^4$$

6.4 Best Approximation; Least Squares

There are many applications in which some linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns should be consistent on physical grounds but fails to be so because of measurement errors in the entries of A or $\mathbf{b}$. In such cases one looks for vectors that come as close as possible to being solutions in the sense that they minimize $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean inner product on R^m. In this section we will discuss methods for finding such minimizing vectors.

Least Squares Solutions of Linear Systems

Suppose that $A\mathbf{x} = \mathbf{b}$ is an *inconsistent* linear system of m equations in n unknowns in which we suspect the inconsistency to be caused by errors in the entries of A or $\mathbf{b}$. Since no exact solution is possible, we will look for a vector $\mathbf{x}$ that comes as "close as possible" to being a solution in the sense that it minimizes $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean