Proof of Theorem 7.1.5 Assume that V is an n-dimensional inner product space and that P is the transition matrix from an orthonormal basis B' to an orthonormal basis B. We will denote the norm relative to the inner product on V by the symbol $\|\ \|_V$ to distinguish it from the norm relative to the Euclidean inner product on R^n, which we will denote by $\|\ \|$.

> Recall that $(\mathbf{u})_S$ denotes a coordinate vector expressed in comma-delimited form whereas $[\mathbf{u}]_S$ denotes a coordinate vector expressed in column form.

To prove that P is orthogonal, we will use Theorem 7.1.3 and show that $\|P\mathbf{x}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x}$ in R^n. As a first step in this direction, recall from Theorem 7.1.4(a) that for any orthonormal basis for V the norm of any vector $\mathbf{u}$ in V is the same as the norm of its coordinate vector with respect to the Euclidean inner product, that is,

$$\|\mathbf{u}\|_V = \|[\mathbf{u}]_{B'}\| = \|[\mathbf{u}]_B\|$$

or

$$\|\mathbf{u}\|_V = \|[\mathbf{u}]_{B'}\| = \|P[\mathbf{u}]_{B'}\| \tag{6}$$

Now let $\mathbf{x}$ be any vector in R^n, and let $\mathbf{u}$ be the vector in V whose coordinate vector with respect to the basis B' is $\mathbf{x}$, that is, $[\mathbf{u}]_{B'} = \mathbf{x}$. Thus, from (6),

$$\|\mathbf{u}\| = \|\mathbf{x}\| = \|P\mathbf{x}\|$$

which proves that P is orthogonal. ◀

Exercise Set 7.1

▶ In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse. ◀

1. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4. (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

▶ In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating A^TA, then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1. ◀

5. $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$ **6.** $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

7. Let $T_A: R^3 \to R^3$ be multiplication by the orthogonal matrix in Exercise 5. Find $T_A(\mathbf{x})$ for the vector $\mathbf{x} = (-2, 3, 5)$, and confirm that $\|T_A(\mathbf{x})\| = \|\mathbf{x}\|$ relative to the Euclidean inner product on R^3.

8. Let $T_A: R^3 \to R^3$ be multiplication by the orthogonal matrix in Exercise 6. Find $T_A(\mathbf{x})$ for the vector $\mathbf{x} = (0, 1, 4)$, and confirm $\|T_A(\mathbf{x})\| = \|\mathbf{x}\|$ relative to the Euclidean inner product on R^3.

9. Are the standard matrices for the reflections in Tables 1 and 2 of Section 4.9 orthogonal?

10. Are the standard matrices for the orthogonal projections in Tables 3 and 4 of Section 4.9 orthogonal?

11. What conditions must a and b satisfy for the matrix

$$\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$$

to be orthogonal?

12. Under what conditions will a diagonal matrix be orthogonal?

13. Let a rectangular $x'y'$-coordinate system be obtained by rotating a rectangular xy-coordinate system counterclockwise through the angle $\theta = \pi/3$.

(a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2, 6)$.

(b) Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5, 2)$.

14. Repeat Exercise 13 with $\theta = 3\pi/4$.

15. Let a rectangular $x'y'z'$-coordinate system be obtained by rotating a rectangular xyz-coordinate system counterclockwise about the z-axis (looking down the z-axis) through the angle $\theta = \pi/4$.

(a) Find the $x'y'z'$-coordinates of the point whose xyz-coordinates are $(-1, 2, 5)$.

(b) Find the xyz-coordinates of the point whose $x'y'z'$-coordinates are $(1, 6, -3)$.

16. Repeat Exercise 15 for a rotation of $\theta = 3\pi/4$ counterclockwise about the x-axis (looking along the positive x-axis toward the origin).

17. Repeat Exercise 15 for a rotation of $\theta = \pi/3$ counterclockwise about the y-axis (looking along the positive y-axis toward the origin).

18. A rectangular $x'y'z'$-coordinate system is obtained by rotating an xyz-coordinate system counterclockwise about the y-axis through an angle θ (looking along the positive y-axis toward the origin). Find a matrix A such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where (x, y, z) and (x', y', z') are the coordinates of the same point in the xyz- and $x'y'z'$-systems, respectively.

19. Repeat Exercise 18 for a rotation about the x-axis.

20. A rectangular $x''y''z''$-coordinate system is obtained by first rotating a rectangular xyz-coordinate system 60° counterclockwise about the z-axis (looking down the positive z-axis) to obtain an $x'y'z'$-coordinate system, and then rotating the $x'y'z'$-coordinate system 45° counterclockwise about the y'-axis (looking along the positive y'-axis toward the origin). Find a matrix A such that

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where (x, y, z) and (x'', y'', z'') are the xyz- and $x''y''z''$-coordinates of the same point.

21. A linear operator on R^2 is called ***rigid*** if it does not change the lengths of vectors, and it is called ***angle preserving*** if it does not change the angle between nonzero vectors.

(a) Identify two different types of linear operators that are rigid.

(b) Identify two different types of linear operators that are angle preserving.

(c) Are there any linear operators on R^2 that are rigid and not angle preserving? Angle preserving and not rigid? Justify your answer.

22. Can an orthogonal operator $T_A: R^n \to R^n$ map nonzero vectors that are not orthogonal into orthogonal vectors? Justify your answer.

23. The set $S = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}}\right\}$ is an orthonormal basis for P_2 with respect to the evaluation inner product at the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Let $\mathbf{p} = p(x) = 1 + x + x^2$ and $\mathbf{q} = q(x) = 2x - x^2$.

(a) Find $(\mathbf{p})_S$ and $(\mathbf{q})_S$.

(b) Use Theorem 7.1.4 to compute $\|\mathbf{p}\|$, $d(\mathbf{p}, \mathbf{q})$ and $\langle \mathbf{p}, \mathbf{q} \rangle$.

24. The sets $S = \{1, x\}$ and $S' = \left\{\frac{1}{\sqrt{2}}(1 + x), \frac{1}{\sqrt{2}}(1 - x)\right\}$ are orthonormal bases for P_1 with respect to the standard inner product. Find the transition matrix P from S to S', and verify that the conclusion of Theorem 7.1.5 holds for P.

Working with Proofs

25. Prove that if $\mathbf{x}$ is an $n \times 1$ matrix, then the matrix

$$A = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T$$

is both orthogonal and symmetric.

26. Prove that a 2×2 orthogonal matrix A has only one of two possible forms:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 \le \theta < 2\pi$. [*Hint:* Start with a general 2×2 matrix A, and use the fact that the column vectors form an orthonormal set in R^2.]

27. (a) Use the result in Exercise 26 to prove that multiplication by a 2×2 orthogonal matrix is a rotation if $\det(A) = 1$ and a reflection followed by a rotation if $\det(A) = -1$.

(b) In the case where the transformation in part (a) is a reflection followed by a rotation, show that the same transformation can be accomplished by a single reflection about an appropriate line through the origin. What is that line? [*Hint:* See Formula (6) of Section 4.9.]

28. In each part, use the result in Exercise 27(a) to determine whether multiplication by A is a rotation or a reflection followed by rotation. Find the angle of rotation in both cases, and in the case where it is a reflection followed by a rotation find an equation for the line through the origin referenced in Exercise 27(b).

(a) $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

29. The result in Exercise 27(a) has an analog for 3×3 orthogonal matrices. It can be proved that multiplication by a 3×3 orthogonal matrix A is a rotation about some line through the origin of R^3 if $\det(A) = 1$ and is a reflection about some coordinate plane followed by a rotation about some line through the origin if $\det(A) = -1$. Use the first of these facts and Theorem 7.1.2 to prove that any composition of rotations about lines through the origin in R^3 can be accomplished by a single rotation about an appropriate line through the origin.

30. ***Euler's Axis of Rotation Theorem*** states that: *If A is an orthogonal* 3×3 *matrix for which* $\det(A) = 1$, *then multiplication by A is a rotation about a line through the origin in* R^3. *Moreover, if* $\mathbf{u}$ *is a unit vector along this line, then* $A\mathbf{u} = \mathbf{u}$.

(a) Confirm that the following matrix A is orthogonal, that $\det(A) = 1$, and that there is a unit vector $\mathbf{u}$ for which $A\mathbf{u} = \mathbf{u}$.

$$A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

(b) Use Formula (3) of Section 4.9 to prove that if A is a 3×3 orthogonal matrix for which $\det(A) = 1$, then the angle of rotation resulting from multiplication by A satisfies the equation $\cos\theta = \frac{1}{2}[\operatorname{tr}(A) - 1]$. Use this result to find the angle of rotation for the rotation matrix in part (a).

31. Prove the equivalence of statements (a) and (c) that are given in Theorem 7.1.1.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is orthogonal.

(b) The matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ is orthogonal.

(c) An $m \times n$ matrix A is orthogonal if $A^TA = I$.

(d) A square matrix whose columns form an orthogonal set is orthogonal.

(e) Every orthogonal matrix is invertible.

(f) If A is an orthogonal matrix, then A^2 is orthogonal and $(\det A)^2 = 1$.

(g) Every eigenvalue of an orthogonal matrix has absolute value 1.

(h) If A is a square matrix and $\|A\mathbf{u}\| = 1$ for all unit vectors $\mathbf{u}$, then A is orthogonal.

Working with Technology

T1. If $\mathbf{a}$ is a nonzero vector in R^n, then $\mathbf{a}\mathbf{a}^T$ is called the ***outer product*** of $\mathbf{a}$ with itself, the subspace $\mathbf{a}^\perp$ is called the ***hyperplane*** in R^n orthogonal to $\mathbf{a}$, and the $n \times n$ orthogonal matrix

$$H_{\mathbf{a}^\perp} = I - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T$$

is called the ***Householder matrix*** or the ***Householder reflection*** about $\mathbf{a}^\perp$, named in honor of the American mathematician Alston S. Householder (1904–1993). In R^2 the matrix $H_{\mathbf{a}^\perp}$ represents a reflection about the line through the origin that is orthogonal to $\mathbf{a}$, and in R^3 it represents a reflection about the plane through the origin that is orthogonal to $\mathbf{a}$. In higher dimensions we can view $H_{\mathbf{a}^\perp}$ as a "reflection" about the hyperplane $\mathbf{a}^\perp$. Householder reflections are important in large-scale implementations of numerical algorithms, particularly QR-decompositions, because they can be used to transform a given vector into a vector with specified zero components while leaving the other components unchanged. This is a consequence of the following theorem [see *Contemporary Linear Algebra*, by Howard Anton and Robert C. Busby (Hoboken, NJ: John Wiley & Sons, 2003, p. 422)].

> ***Theorem.*** *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are distinct vectors in* R^n *with the same norm, then the Householder reflection about the hyperplane* $(\mathbf{v} - \mathbf{w})^\perp$ *maps* $\mathbf{v}$ *into* $\mathbf{w}$ *and conversely.*

(a) Find a Householder reflection that maps the vector $\mathbf{v} = (4, 2, 4)$ into a vector $\mathbf{w}$ that has zeros as its second and third components. Find $\mathbf{w}$.

(b) Find a Householder reflection that maps the vector $\mathbf{v} = (3, 4, 2, 4)$ into the vector whose last two entries are zero, while leaving the first entry unchanged. Find $\mathbf{w}$.

7.2 Orthogonal Diagonalization

In this section we will be concerned with the problem of diagonalizing a symmetric matrix A. As we will see, this problem is closely related to that of finding an orthonormal basis for R^n that consists of eigenvectors of A. Problems of this type are important because many of the matrices that arise in applications are symmetric.

The Orthogonal Diagonalization Problem

In Section 5.2 we defined two square matrices, A and B, to be *similar* if there is an *invertible* matrix P such that $P^{-1}AP = B$. In this section we will be concerned with the special case in which it is possible to find an *orthogonal* matrix P for which this relationship holds.

We begin with the following definition.