Remark In many numerical algorithms the initial matrix is first converted to upper Hessenberg form to reduce the amount of computation in subsequent parts of the algorithm. Many computer packages have built-in commands for finding Schur and Hessenberg decompositions.

Exercise Set 7.2

In Exercises 1–6, find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces.

1. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

10. $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 15–18, find the spectral decomposition of the matrix.

15. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

17. $\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

In Exercises 19–20, determine whether there exists a 3×3 symmetric matrix whose eigenvalues are $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 7$ and for which the corresponding eigenvectors are as stated. If there is such a matrix, find it, and if there is none, explain why not.

19. $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

20. $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

21. Let A be a diagonalizable matrix with the property that eigenvectors corresponding to distinct eigenvalues are orthogonal. Must A be symmetric? Explain your reasoning.

22. Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

23. Let $T_A: R^2 \to R^2$ be multiplication by A. Find two orthogonal unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ are orthogonal.

(a) $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

24. Let $T_A: R^3 \to R^3$ be multiplication by A. Find two orthogonal unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ are orthogonal.

(a) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Working with Proofs

25. Prove that if A is any $m \times n$ matrix, then A^TA has an orthonormal set of n eigenvectors.

26. Prove: If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for R^n, and if A can be expressed as

$$A = c_1\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + c_n\mathbf{u}_n\mathbf{u}_n^T$$

then A is symmetric and has eigenvalues $c_1, c_2, \ldots, c_n$.

27. Use the result in Exercise 29 of Section 5.1 to prove Theorem 7.2.2(*a*) for 2×2 symmetric matrices.

28. (a) Prove that if $\mathbf{v}$ is any $n \times 1$ matrix and I is the $n \times n$ identity matrix, then $I - \mathbf{v}\mathbf{v}^T$ is orthogonally diagonalizable.

(b) Find a matrix P that orthogonally diagonalizes $I - \mathbf{v}\mathbf{v}^T$ if

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

29. Prove that if A is a symmetric orthogonal matrix, then 1 and -1 are the only possible eigenvalues.

30. Is the converse of Theorem 7.2.2(*b*) true? Justify your answer.

31. In this exercise we will show that a symmetric matrix A is orthogonally diagonalizable, thereby completing the missing part of Theorem 7.2.1. We will proceed in two steps: first we will show that A is diagonalizable, and then we will build on that result to show that A is orthogonally diagonalizable.

(a) Assume that A is a symmetric $n \times n$ matrix. One way to prove that A is diagonalizable is to show that for each eigenvalue λ_0 the geometric multiplicity is equal to the algebraic multiplicity. For this purpose, assume that the geometric multiplicity of λ_0 is k, let $B_0 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for the eigenspace corresponding to the eigenvalue λ_0, extend this to an orthonormal basis $B_0 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n, and let P be the matrix having the vectors of B as columns. As shown in Exercise 40(b) of Section 5.2, the product AP can be written as

$$AP = P\begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

Use the fact that B is an orthonormal basis to prove that $X = 0$ [a zero matrix of size $n \times (n-k)$].

(b) It follows from part (a) and Exercise 40(c) of Section 5.2 that A has the same characteristic polynomial as

$$C = P\begin{bmatrix} \lambda_0 I_k & 0 \\ 0 & Y \end{bmatrix}$$

Use this fact and Exercise 40(d) of Section 5.2 to prove that the algebraic multiplicity of λ_0 is the same as the geometric multiplicity of λ_0. This establishes that A is diagonalizable.

(c) Use Theorem 7.2.2(*b*) and the fact that A is diagonalizable to prove that A is orthogonally diagonalizable.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix, then AA^T and A^TA are orthogonally diagonalizable.

(b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors from distinct eigenspaces of a symmetric matrix with real entries, then

$$\|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2$$

(c) Every orthogonal matrix is orthogonally diagonalizable.

(d) If A is both invertible and orthogonally diagonalizable, then A^{-1} is orthogonally diagonalizable.

(e) Every eigenvalue of an orthogonal matrix has absolute value 1.

(f) If A is an $n \times n$ orthogonally diagonalizable matrix, then there exists an orthonormal basis for R^n consisting of eigenvectors of A.

(g) If A is orthogonally diagonalizable, then A has real eigenvalues.

Working with Technology

T1. If your technology utility has an orthogonal diagonalization capability, use it to confirm the final result obtained in Example 1.

T2. For the given matrix A, find orthonormal bases for the eigenspaces of A, and use those basis vectors to construct an orthogonal matrix P for which P^TAP is diagonal.

$$A = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{bmatrix}$$

T3. Find a spectral decomposition of the matrix A in Exercise T2.

7.3 Quadratic Forms

In this section we will use matrix methods to study real-valued functions of several variables in which each term is either the square of a variable or the product of two variables. Such functions arise in a variety of applications, including geometry, vibrations of mechanical systems, statistics, and electrical engineering.

Definition of a Quadratic Form

Expressions of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

occurred in our study of linear equations and linear systems. If $a_1, a_2, \ldots, a_n$ are treated as fixed constants, then this expression is a real-valued function of the n ***variables*** $x_1, x_2, \ldots, x_n$ and is called a ***linear form*** on R^n. All variables in a linear form occur to