Exercise Set 8.1

▶ In Exercises 1–2, suppose that T is a mapping whose domain is the vector space M_{22}. In each part, determine whether T is a linear transformation, and if so, find its kernel. ◀

1. (a) $T(A) = A^2$ (b) $T(A) = \text{tr}(A)$

(c) $T(A) = A + A^T$

2. (a) $T(A) = (A)_{11}$ (b) $T(A) = 0_{2\times 2}$

(c) $T(A) = cA$

▶ In Exercises 3–9, determine whether the mapping T is a linear transformation, and if so, find its kernel. ◀

3. $T: R^3 \to R$, where $T(\mathbf{u}) = \|\mathbf{u}\|$.

4. $T: R^3 \to R^3$, where $\mathbf{v}_0$ is a fixed vector in R^3 and $T(\mathbf{u}) = \mathbf{u} \times \mathbf{v}_0$.

5. $T: M_{22} \to M_{23}$, where B is a fixed 2×3 matrix and $T(A) = AB$.

6. $T: M_{22} \to R$, where

(a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

(b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$

7. $T: P_2 \to P_2$, where

(a) $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$

(b) $T(a_0 + a_1x + a_2x^2)$
$= (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$

8. $T: F(-\infty, \infty) \to F(-\infty, \infty)$, where

(a) $T(f(x)) = 1 + f(x)$ (b) $T(f(x)) = f(x+1)$

9. $T: R^\infty \to R^\infty$, where
$T(a_0, a_1, a_2, \ldots, a_n, \ldots) = (0, a_0, a_1, a_2, \ldots, a_n, \ldots)$

10. Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

(a) x^2 (b) 0 (c) $1 + x$ (d) $-x$

11. Let $T: P_2 \to P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

(a) $x + x^2$ (b) $1 + x$ (c) $3 - x^2$ (d) $-x$

12. Let V be any vector space, and let $T: V \to V$ be defined by $T(\mathbf{v}) = 3\mathbf{v}$.

(a) What is the kernel of T?

(b) What is the range of T?

13. In each part, use the given information to find the nullity of the linear transformation T.

(a) $T: R^5 \to P_5$ has rank 3.

(b) $T: P_4 \to P_3$ has rank 1.

(c) The range of $T: M_{mn} \to R^3$ is R^3.

(d) $T: M_{22} \to M_{22}$ has rank 3.

14. In each part, use the given information to find the rank of the linear transformation T.

(a) $T: R^7 \to M_{32}$ has nullity 2.

(b) $T: P_3 \to R$ has nullity 1.

(c) The null space of $T: P_5 \to P_5$ is P_5.

(d) $T: P_n \to M_{mn}$ has nullity 3.

15. Let $T: M_{22} \to M_{22}$ be the dilation operator with factor $k = 3$.

(a) Find $T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right)$.

(b) Find the rank and nullity of T.

16. Let $T: P_2 \to P_2$ be the contraction operator with factor $k = 1/4$.

(a) Find $T(1 + 4x + 8x^2)$.

(b) Find the rank and nullity of T.

17. Let $T: P_2 \to R^3$ be the evaluation transformation at the sequence of points $-1, 0, 1$. Find

(a) $T(x^2)$ (b) $\ker(T)$ (c) $R(T)$

18. Let V be the subspace of $C[0, 2\pi]$ spanned by the vectors 1, $\sin x$, and $\cos x$, and let $T: V \to R^3$ be the evaluation transformation at the sequence of points $0, \pi, 2\pi$. Find

(a) $T(1 + \sin x + \cos x)$ (b) $\ker(T)$

(c) $R(T)$

19. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 0)$, and let $T: R^2 \to R^2$ be the linear operator for which

$$T(\mathbf{v}_1) = (1, -2) \quad \text{and} \quad T(\mathbf{v}_2) = (-4, 1)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(5, -3)$.

20. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T: R^2 \to R^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = (-1, 2, 0) \quad \text{and} \quad T(\mathbf{v}_2) = (0, -3, 5)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(2, -3)$.

21. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 0)$, and let $T: R^3 \to R^3$ be the linear operator for which

$$T(\mathbf{v}_1) = (2, -1, 4), \quad T(\mathbf{v}_2) = (3, 0, 1), \quad T(\mathbf{v}_3) = (-1, 5, 1)$$

Find a formula for $T(x_1, x_2, x_3)$, and use that formula to find $T(2, 4, -1)$.

22. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$, and let $T: R^3 \to R^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (-1, 1), \quad T(\mathbf{v}_3) = (0, 1)$$

Find a formula for $T(x_1, x_2, x_3)$, and use that formula to find $T(7, 13, 7)$.

23. Let $T: P_3 \to P_2$ be the mapping defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = 5a_0 + a_3x^2$$

(a) Show that T is linear.

(b) Find a basis for the kernel of T.

(c) Find a basis for the range of T.

24. Let $T: P_2 \to P_2$ be the mapping defined by

$$T(a_0 + a_1x + a_2x^2) = 3a_0 + a_1x + (a_0 + a_1)x^2$$

(a) Show that T is linear.

(b) Find a basis for the kernel of T.

(c) Find a basis for the range of T.

25. (a) ***(Calculus required)*** Let $D: P_3 \to P_2$ be the differentiation transformation $D(\mathbf{p}) = p'(x)$. What is the kernel of D?

(b) ***(Calculus required)*** Let $J: P_1 \to R$ be the integration transformation $J(\mathbf{p}) = \int_{-1}^{1} p(x)\,dx$. What is the kernel of J?

26. ***(Calculus required)*** Let $V = C[a, b]$ be the vector space of continuous functions on $[a, b]$, and let $T: V \to V$ be the transformation defined by

$$T(\mathbf{f}) = 5f(x) + 3\int_a^x f(t)\,dt$$

Is T a linear operator?

27. ***(Calculus required)*** Let V be the vector space of real-valued functions with continuous derivatives of all orders on the interval $(-\infty, \infty)$, and let $W = F(-\infty, \infty)$ be the vector space of real-valued functions defined on $(-\infty, \infty)$.

(a) Find a linear transformation $T: V \to W$ whose kernel is P_3.

(b) Find a linear transformation $T: V \to W$ whose kernel is P_n.

28. For a positive integer $n > 1$, let $T: M_{nn} \to R$ be the linear transformation defined by $T(A) = \text{tr}(A)$, where A is an $n \times n$ matrix with real entries. Determine the dimension of $\ker(T)$.

29. (a) Let $T: V \to R^3$ be a linear transformation from a vector space V to R^3. Geometrically, what are the possibilities for the range of T?

(b) Let $T: R^3 \to W$ be a linear transformation from R^3 to a vector space W. Geometrically, what are the possibilities for the kernel of T?

30. In each part, determine whether the mapping $T: P_n \to P_n$ is linear.

(a) $T(p(x)) = p(x + 1)$

(b) $T(p(x)) = p(x) + 1$

31. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V, and let $T: V \to R^3$ be a linear transformation for which

$$T(\mathbf{v}_1) = (1, -1, 2), \quad T(\mathbf{v}_2) = (0, 3, 2), \quad T(\mathbf{v}_3) = (-3, 1, 2)$$

Find $T(2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3)$.

Working with Proofs

32. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T: V \to W$ be a linear transformation. Prove that if

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0}$$

then T is the zero transformation.

33. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T: V \to V$ be a linear operator. Prove that if

$$T(\mathbf{v}_1) = \mathbf{v}_1, \quad T(\mathbf{v}_2) = \mathbf{v}_2, \ldots, \quad T(\mathbf{v}_n) = \mathbf{v}_n$$

then T is the identity transformation on V.

34. Prove: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ are vectors in a vector space W, not necessarily distinct, then there exists a linear transformation $T: V \to W$ such that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \quad T(\mathbf{v}_2) = \mathbf{w}_2, \ldots, \quad T(\mathbf{v}_n) = \mathbf{w}_n$$

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V and all scalars c_1 and c_2, then T is a linear transformation.

(b) If $\mathbf{v}$ is a nonzero vector in V, then there is exactly one linear transformation $T: V \to W$ such that $T(-\mathbf{v}) = -T(\mathbf{v})$.

(c) There is exactly one linear transformation $T: V \to W$ for which $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} - \mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V.

(d) If $\mathbf{v}_0$ is a nonzero vector in V, then the formula $T(\mathbf{v}) = \mathbf{v}_0 + \mathbf{v}$ defines a linear operator on V.

(e) The kernel of a linear transformation is a vector space.

(f) The range of a linear transformation is a vector space.

(g) If $T: P_6 \to M_{22}$ is a linear transformation, then the nullity of T is 3.

(h) The function $T: M_{22} \to R$ defined by $T(A) = \det A$ is a linear transformation.

(i) The linear transformation $T: M_{22} \to M_{22}$ defined by

$$T(A) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} A$$

has rank 1.