Now, taking $T_2^{-1}$ of each side of this equation, then taking $T_1^{-1}$ of each side of the result, and then using (4) yields (verify)

$$\mathbf{u} = T_1^{-1}(T_2^{-1}(\mathbf{w}))$$

or, equivalently,

$$\mathbf{u} = (T_1^{-1} \circ T_2^{-1})(\mathbf{w})$$

In words, part (*b*) of Theorem 8.2.4 states that *the inverse of a composition is the composition of the inverses in the reverse order*. This result can be extended to compositions of three or more linear transformations; for example,

$$(T_3 \circ T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1} \circ T_3^{-1} \tag{8}$$

Note the order of the subscripts on the two sides of Formula (9).

In the case where $T_A$, $T_B$, and $T_C$ are matrix operators on $R^n$, Formula (8) can be written as

$$(T_C \circ T_B \circ T_A)^{-1} = T_A^{-1} \circ T_B^{-1} \circ T_C^{-1}$$

or alternatively as

$$(T_{CBA})^{-1} = T_{A^{-1}B^{-1}C^{-1}} \tag{9}$$



## Exercise Set 8.2

In Exercises 1–2, determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

**1.** (a) $T: R^2 \to R^2$, where $T(x, y) = (y, x)$

(b) $T: R^2 \to R^3$, where $T(x, y) = (x, y, x + y)$

(c) $T: R^3 \to R^2$, where $T(x, y, z) = (x + y + z, x - y - z)$

**2.** (a) $T: R^2 \to R^3$, where $T(x, y) = (x - y, y - x, 2x - 2y)$

(b) $T: R^2 \to R^2$, where $T(x, y) = (0, 2x + 3y)$

(c) $T: R^2 \to R^2$, where $T(x, y) = (x + y, x - y)$

In Exercises 3–4, determine whether multiplication by $A$ is one-to-one by computing the nullity of $A$ and then applying Theorem 8.2.1.

**3.** (a) $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

**4.** (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \\ 3 & 9 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -3 & 6 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**5.** Use the given information to determine whether the linear transformation is one-to-one.

(a) $T: V \to W$; $\text{nullity}(T) = 0$

(b) $T: V \to W$; $\text{rank}(T) = \dim(V)$

(c) $T: V \to W$; $\dim(W) < \dim(V)$

**6.** Use the given information to determine whether the linear operator is one-to-one, onto, both, or neither.

(a) $T: V \to V$; $\text{nullity}(T) = 0$

(b) $T: V \to V$; $\text{rank}(T) < \dim(V)$

(c) $T: V \to V$; $R(T) = V$

**7.** Show that the linear transformation $T: P_2 \to R^2$ defined by $T(p(x)) = (p(-1), p(1))$ is not one-to-one by finding a nonzero polynomial that maps into $\mathbf{0} = (0, 0)$. Do you think that this transformation is onto?

**8.** Show that the linear transformation $T: P_2 \to P_2$ defined by $T(p(x)) = p(x + 1)$ is one-to-one. Do you think that this transformation is onto?

**9.** Let $\mathbf{a}$ be a fixed vector in $R^3$. Does the formula $T(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$ define a one-to-one linear operator on $R^3$? Explain your reasoning.

**10.** Let $E$ be a fixed $2 \times 2$ elementary matrix. Does the formula $T(A) = EA$ define a one-to-one linear operator on $M_{22}$? Explain your reasoning.

In Exercises 11–12, compute $(T_2 \circ T_1)(x, y)$.

**11.** $T_1(x, y) = (2x, 3y)$, $T_2(x, y) = (x - y, x + y)$

**12.** $T_1(x, y) = (2x, -3y, x + y)$,
$T_2(x, y, z) = (x - y, y + z)$

► In Exercises 13–14, compute $(T_3 \circ T_2 \circ T_1)(x, y)$. ◄

**13.** $T_1(x, y) = (-2y, 3x, x - 2y)$, $T_2(x, y, z) = (y, z, x)$,
$T_3(x, y, z) = (x + z, y - z)$

**14.** $T_1(x, y) = (x + y, y, -x)$,
$T_2(x, y, z) = (0, x + y + z, 3y)$,
$T_3(x, y, z) = (3x + 2y, 4z - x - 3y)$

**15.** Let $T_1: M_{22} \to R$ and $T_2: M_{22} \to M_{22}$ be the linear transformations given by $T_1(A) = \text{tr}(A)$ and $T_2(A) = A^T$.

(a) Find $(T_1 \circ T_2)(A)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(b) Can you find $(T_2 \circ T_1)(A)$? Explain.

**16.** Rework Exercise 15 given that $T_1: M_{22} \to M_{22}$ and $T_2: M_{22} \to M_{22}$ are the linear transformations, $T_1(A) = kA$ and $T_2(A) = A^T$, where $k$ is a scalar.

**17.** Suppose that the linear transformations $T_1: P_2 \to P_2$ and $T_2: P_2 \to P_3$ are given by the formulas $T_1(p(x)) = p(x + 1)$ and $T_2(p(x)) = xp(x)$. Find $(T_2 \circ T_1)(a_0 + a_1x + a_2x^2)$.

**18.** Let $T_1: P_n \to P_n$ and $T_2: P_n \to P_n$ be the linear operators given by $T_1(p(x)) = p(x - 1)$ and $T_2(p(x)) = p(x + 1)$. Find $(T_1 \circ T_2)(p(x))$ and $(T_2 \circ T_1)(p(x))$.

**19.** Let $T: P_1 \to R^2$ be the function defined by the formula

$$T(p(x)) = (p(0), p(1))$$

(a) Find $T(1 - 2x)$.

(b) Show that $T$ is a linear transformation.

(c) Show that $T$ is one-to-one.

(d) Find $T^{-1}(2, 3)$, and sketch its graph.

**20.** In each part, determine whether the linear operator $T: R^n \to R^n$ is one-to-one; if so, find $T^{-1}(x_1, x_2, \ldots, x_n)$.

(a) $T(x_1, x_2, \ldots, x_n) = (0, x_1, x_2, \ldots, x_{n-1})$

(b) $T(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_2, x_1)$

(c) $T(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1)$

**21.** Let $T: R^n \to R^n$ be the linear operator defined by the formula

$$T(x_1, x_2, \ldots, x_n) = (a_1x_1, a_2x_2, \ldots, a_nx_n)$$

where $a_1, \ldots, a_n$ are constants.

(a) Under what conditions will $T$ have an inverse?

(b) Assuming that the conditions determined in part (a) are satisfied, find a formula for $T^{-1}(x_1, x_2, \ldots, x_n)$.

**22.** Let $T_1: R^2 \to R^2$ and $T_2: R^2 \to R^2$ be the linear operators given by the formulas

$$T_1(x, y) = (x + y, x - y) \quad \text{and} \quad T_2(x, y) = (2x + y, x - 2y)$$

(a) Show that $T_1$ and $T_2$ are one-to-one.

(b) Find formulas for

$$T_1^{-1}(x, y), \quad T_2^{-1}(x, y), \quad (T_2 \circ T_1)^{-1}(x, y)$$

(c) Verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

**23.** Let $T_1: P_2 \to P_3$ and $T_2: P_3 \to P_3$ be the linear transformations given by the formulas

$$T_1\big(p(x)\big) = xp(x) \quad \text{and} \quad T_2\big(p(x)\big) = p(x + 1)$$

(a) Find formulas for $T_1^{-1}\big(p(x)\big)$, $T_2^{-1}\big(p(x)\big)$, and $(T_1^{-1} \circ T_2^{-1})\big(p(x)\big)$.

(b) Verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

**24.** Let $T_A: R^3 \to R^3$, $T_B: R^3 \to R^3$, and $T_C: R^3 \to R^3$ be the reflections about the $xy$-plane, the $xz$-plane, and the $yz$-plane, respectively. Verify Formula (9) for these linear operators.

**25.** Let $T_1: V \to V$ be the dilation $T_1(\mathbf{v}) = 4\mathbf{v}$. Find a linear operator $T_2: V \to V$ such that $T_1 \circ T_2 = I$ and $T_2 \circ T_1 = I$.

**26.** Let $T_1: M_{22} \to P_1$ and $T_2: P_1 \to R^3$ be the linear transformations given by $T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b) + (c + d)x$ and $T_2(a + bx) = (a, b, a)$.

(a) Find the formula for $T_2 \circ T_1$.

(b) Show that $T_2 \circ T_1$ is not one-to-one by finding distinct $2 \times 2$ matrices $A$ and $B$ such that

$$(T_2 \circ T_1)(A) = (T_2 \circ T_1)(B)$$

(c) Show that $T_2 \circ T_1$ is not onto by finding a vector $(a, b, c)$ in $R^3$ that is not in the range of $T_2 \circ T_1$.

**27.** Let $T: R^3 \to R^3$ be the orthogonal projection of $R^3$ onto the $xy$-plane. Show that $T \circ T = T$.

**28.** (***Calculus required***) Let $V$ be the vector space $C^1[0, 1]$ and let $T: V \to R$ be defined by

$$T(\mathbf{f}) = f(0) + 2f'(0) + 3f'(1)$$

Verify that $T$ is a linear transformation. Determine whether $T$ is one-to-one, and justify your conclusion.

**29.** (***Calculus required***) The Fundamental Theorem of Calculus implies that integration and differentiation reverse the actions of each other. Define a transformation $D: P_n \to P_{n-1}$ by $D(p(x)) = p'(x)$, and define $J: P_{n-1} \to P_n$ by

$$J(p(x)) = \int_0^x p(t)\,dt$$

(a) Show that $D$ and $J$ are linear transformations.

(b) Explain why $J$ is not the inverse transformation of $D$.

(c) Can the domains and/or codomains of $D$ and $J$ be restricted so they are inverse linear transformations?

**30.** **(*Calculus required*)** Let

$$D(\mathbf{f}) = f'(x) \quad \text{and} \quad J(\mathbf{f}) = \int_0^x f(t)\,dt$$

be the linear transformations in Examples 11 and 12 of Section 8.1. Find $(J \circ D)(\mathbf{f})$ for

(a) $\mathbf{f}(x) = x^2 + 3x + 2$. (b) $\mathbf{f}(x) = \sin x$.

**31.** **(*Calculus required*)** Let $J: P_1 \to R$ be the integration transformation $J(\mathbf{p}) = \int_{-1}^{1} p(x)dx$. Determine whether $J$ is one-to-one. Justify your answer.

**32.** **(*Calculus required*)** Let $D: P_n \to P_{n-1}$ be the differentiation transformation $D(p(x)) = p'(x)$. Determine whether $D$ is onto, and justify your answer.

### Working with Proofs

**33.** Prove: If $T: V \to W$ is a one-to-one linear transformation, then $T^{-1}: R(T) \to V$ is a one-to-one linear transformation.

**34.** Use the definition of $T_3 \circ T_2 \circ T_1$ given by Formula (3) to prove that

(a) $T_3 \circ T_2 \circ T_1$ is a linear transformation.

(b) $T_3 \circ T_2 \circ T_1 = (T_3 \circ T_2) \circ T_1$.

(c) $T_3 \circ T_2 \circ T_1 = T_3 \circ (T_2 \circ T_1)$.

**35.** Let $q_0(x)$ be a fixed polynomial of degree $m$, and define a function $T$ with domain $P_n$ by the formula $T(p(x)) = p(q_0(x))$. Prove that $T$ is a linear transformation.

**36.** Prove: If there exists an onto linear transformation $T: V \to W$ then $\dim(V) \geq \dim(W)$.

**37.** Prove: If $V$ and $W$ are finite-dimensional vector spaces such that $\dim(W) < \dim(V)$, then there is no one-to-one linear transformation $T: V \to W$.

### True-False Exercises

**TF.** In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) The composition of two linear transformations is also a linear transformation.

(b) If $T_1: V \to V$ and $T_2: V \to V$ are any two linear operators, then $T_1 \circ T_2 = T_2 \circ T_1$.

(c) The inverse of a one-to-one linear transformation is a linear transformation.

(d) If a linear transformation $T$ has an inverse, then the kernel of $T$ is the zero subspace.

(e) If $T: R^2 \to R^2$ is the orthogonal projection onto the $x$-axis, then $T^{-1}: R^2 \to R^2$ maps each point on the $x$-axis onto a line that is perpendicular to the $x$-axis.

(f) If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, and if $T_1$ is not one-to-one, then neither is $T_2 \circ T_1$.

## 8.3 Isomorphism

In this section we will establish a fundamental connection between real finite-dimensional vector spaces and the Euclidean space $R^n$. This connection is not only important theoretically, but it has practical applications in that is allows us to perform vector computations in general vector spaces by working with the vectors in $R^n$.

### *Isomorphism*

Although many of the theorems in this text have been concerned exclusively with the vector space $R^n$, this is not as limiting as it might seem. We will show that the vector space $R^n$ is the "mother" of all real $n$-dimensional vector spaces in the sense that every $n$-dimensional vector space must have the same algebraic structure as $R^n$ even though its vectors may not be expressed as $n$-tuples. To explain what we mean by this, we will need the following definition.

> **DEFINITION 1** A linear transformation $T: V \to W$ that is both one-to-one and onto is said to be an ***isomorphism***, and $W$ is said to be ***isomorphic*** to $V$.

In the exercises we will ask you to show that if $T: V \to W$ is an isomorphism, then $T^{-1}: W \to V$ is also an isomorphism. Accordingly, we will usually say simply that ***V and W are isomorphic*** and that ***T is an isomorphism between V and W***.

The word *isomorphic* is derived from the Greek words *iso*, meaning "identical," and *morphe*, meaning "form." This terminology is appropriate because, as we will now explain, isomorphic vector spaces have the same "algebraic form," even though they