1. (20 points) Consider the following function

$$f(x) = \frac{x^2 - 10x + 25}{x - 3} = \frac{(x-5)^2}{(x-3)}$$

(a) (10 points) Find the absolute extrema on $[-1, 4]$

(b) (10 points) Determine the concavity on the same interval.

a) $$f' = \frac{2(x-5)(x-3) - (x-5)^2(1)}{(x-3)^2} = \frac{(x-5)\left[2x-6-x+5\right]}{(x-3)^2}$$

$$= \frac{(x-5)(x-1)}{(x-3)^2} = x^2 - 6x + 5$$

Critical #'s:

$x = 5$ ← not in $[-1, 4]$

$x = 1$

$x = 3$ ← not in domain

$f(-1) = (-6)^2/-4 = 36/-4 = -9$ ← $\boxed{\text{abs. min } (-1, -9)}$

$f(1) = (-4)^2/-2 = 16/-2 = -8$

$f(4) = (1)^2/1 = 1$ ← $\boxed{\text{abs. max } (4, 1)}$

b) $$f'' = \frac{(2x-6)(x-3)^2 - (x^2-6x+5)(2)(x-3)}{(x-3)^4}$$

$$= \frac{2(x-3)\left[(x-3)^2 - (x^2-6x+5)\right]}{(x-3)^4}$$

$$= \frac{2\left[x^2 - 6x + 9 - x^2 + 6x - 5\right]}{(x-3)^3}$$

$$= \frac{2(4)}{(x-3)^3} = \frac{8}{(x-3)^3}$$

− +

3

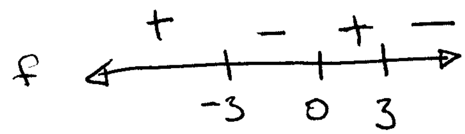
$f''(0) \to -$

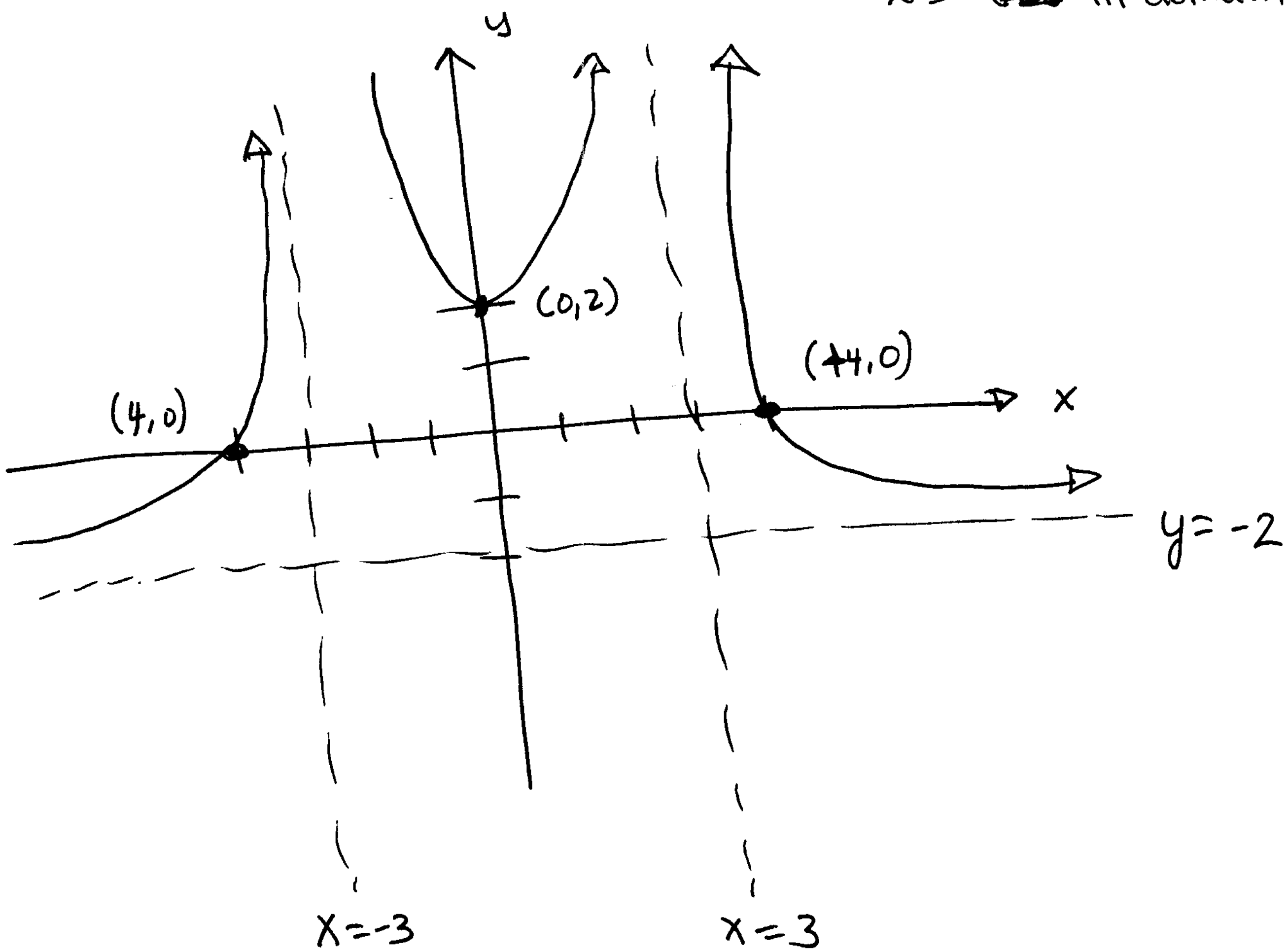
$f''(4) \to +$

$\boxed{\text{Concave down } [-1, 3) \quad \text{Concave up } (3, 4]}$

2. (20 points) Use the information below to sketch the graph of the function. Clearly label important aspects of the graph.

   (a) Intercepts: $(-4,0), (4,0), (0,2)$
   (b) Vertical asymptotes: $x=-3, x=3$
   (c) Horizontal Asymptote: $y=-2$
   (d) $f'(x)<0$ for $(-3,0)$ and $(3,\infty)$
   (e) $f'(x)>0$ for $(-\infty,-3)$ and $(0,3)$
   (f) $f''(x)>0$ for $(-\infty,-3)$, $(-3,3)$ and $(3,\infty)$ → concave up for all x's ~~[illegible]~~ in domain





3. (20 points) Find the x values of the relative extrema for each function. You may leave your answers in radical form, i.e., $x = \sqrt{3}$.

   (a) (8 points) $f(x) = 6x^2 \cdot e^{3x} + 2$

   (b) (12 points) $f(x) = (x^3 - 4x)^{1/3}$

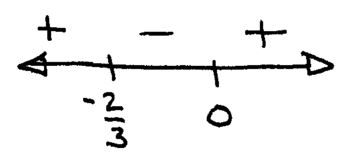
a) $f(x) = 6x^2 e^{3x} + 2$

$f'(x) = 12x e^{3x} + 18x^2 e^{3x}$

$6x e^{3x}(2+3x) = 0$

$6x = 0 \quad e^{3x} \neq 0 \quad 2+3x = 0$

$\underline{x = 0} \qquad \underline{x = -2/3}$



$$\boxed{\begin{array}{ll} x = 0 & \text{rel. min} \\ x = -2/3 & \text{rel max} \end{array}}$$

b) $f(x) = (x^3 - 4x)^{1/3}$

$f'(x) = \frac{1}{3}(x^3-4x)^{-2/3}(3x^2-4) = \dfrac{(3x^2-4)}{3(x^3-4x)^{2/3}}$

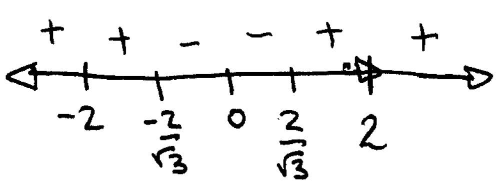
Critical #'s

$f' = 0$: $3x^2 - 4 = 0 \rightarrow x = \pm\sqrt{4/3} = \pm\frac{2}{\sqrt{3}}$

f' DNE : $x^3 - 4x = 0$

$x(x^2-4) = 0 \rightarrow x = 0$

$x = \pm 2$



$$\boxed{\begin{array}{ll} x = \frac{-2}{\sqrt{3}} & \text{rel. max} \\ x = \frac{2}{\sqrt{3}} & \text{rel min} \end{array}}$$

4. (20 points) For the following problems, use the methods presented in the book and class. No credit will be given for guess and check methods.

   (a) (8 points) Find two positive numbers such that their product is 144 and their sum is a minimum.

Constraint: $xy = 144$

Objective: $S = x + y$

$y = \frac{144}{x}$

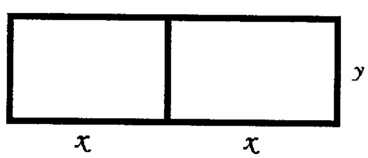
$\rightarrow S = \frac{144}{x} + x$

$\rightarrow S' = -\frac{144}{x^2} + 1 = 0$

$x^2 = 144 \rightarrow x = \pm 12$

$$\boxed{\begin{array}{l} x = 12 \\ y = 12 \end{array}}$$

   (b) (12 points) A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?



Constaint: $200 = 3y + 4x$

Objective: $A = 2xy$

$y = \frac{200 - 4x}{3}$

$\rightarrow A = 2x\left(\frac{200 - 4x}{3}\right)$

$A = \frac{2}{3}(200x - 4x^2)$

$\rightarrow A' = \frac{2}{3}(200 - 8x) = 0$

$8x = 200$

$x = \frac{200}{8}$

$\boxed{x = 25}$

$\rightarrow y = \frac{200 - 4(25)}{3}$

$\boxed{y = \frac{100}{3}}$

(Technically the dimensions will be $50 \times \frac{100}{3}$)

5. (20 points) Short questions:

(a) (4 points) Give a function whose third derivative is equal to 0.

$y = x^2$

(b) (4 points) Which derivative of $f(x) = x^{100}$ will be identically 0, i.e., find n such that $f^{(n)}(x) = 0$.

the 101^{st}

$f^{101}(x) = 0$

(c) (4 points) Give a function in which none of its derivatives (first and higher order) is identically 0.

$f = e^x$, or $f = x^n$ where n is a fraction

(d) (8 points) Find the 2nd derivative of the function below.

$$f(x) = x\ln(3x-1)$$

$$f'(x) = \ln(3x-1) + \frac{3x}{3x-1}$$

$$f''(x) = \frac{3}{3x-1} + \frac{3(3x-1) - (3)(3x)}{(3x-1)^2}$$

EXTRA (5 points) What is the slope of the tangent line of $f(x) = e^x$ as $x \to -\infty$?

$f'(x) = e^x$ ← slope at any x

limit as $x \to -\infty$: $\lim_{x \to -\infty} e^x = 0$

Problem	Max Points	Points
1	20	
2	20	
3	20	
4	20	
5	20	
Extra	5	
Total	100	