1. (6 points) Find the center and radius of the sphere given below.

$$x^2 + y^2 + z^2 - 6x + 4y - 8z - 2 = 0$$

$x^2 - 6x + \underline{9} + y^2 + 4y + \underline{4} + z^2 - 8z + \underline{16} = 2 + 9 + 4 + 16$

$(x-3)^2 + (y+2)^2 + (z-4)^2 = 31$

center: $(3, -2, 4)$

radius: $\sqrt{31}$

2. (9 points) Consider the vectors $\mathbf{u} = \langle 5, -1, 2 \rangle$ and $\mathbf{v} = \langle 1, -3, 0 \rangle$.

(a) (5 points) Calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

$\text{proj}_v u = \dfrac{u \cdot v}{v \cdot v} v$

$u \cdot v = (5)(1) + (-1)(-3) + (2)(0) = 5 + 3 = 8$

$v \cdot v = (1)^2 + (-3)^2 + (0)^2 = 1 + 9 = 10$

$$\boxed{\begin{aligned} &= \tfrac{8}{10} \langle 1, -3, 0 \rangle \\ &= \tfrac{4}{5} \langle 1, -3, 0 \rangle \end{aligned}}$$

(b) (4 points) Find the unit vector of $\mathbf{u}$.

$\cancel{\dfrac{u}{\|u\|}} \; \dfrac{u}{|u|} = \dfrac{\langle 5, -1, 2 \rangle}{\sqrt{5^2 + (-1)^2 + (2)^2}}$

$= \dfrac{\langle 5, -1, 2 \rangle}{\sqrt{25 + 1 + 4}} = \boxed{\dfrac{\langle 5, -1, 2 \rangle}{\sqrt{30}}}$

3. (15 points) The vector-valued equation of velocity for a strange object is given as

$$\mathbf{v}(t) = \left\langle \frac{4}{t-1}, e^{t-2}, \pi \sin(\pi t) \right\rangle \qquad t > 1$$

(a) (6 points) Find the object's vector-valued equation of acceleration.

$$\vec{a}(t) = \vec{v}\,'(t) = \left\langle \frac{-4}{(t-1)^2},\ e^{t-2},\ \pi^2 \cos(\pi t) \right\rangle$$

(b) (9 points) Find the object's vector-valued equation of motion if we know the object passes through $\mathbf{r}(2) = \langle -5, 3, -2 \rangle$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left\langle \frac{4}{t-1},\ e^{t-2},\ \pi \sin(\pi t) \right\rangle dt$$

$$\vec{r}(t) = \left\langle 4\int \frac{1}{t-1}\,dt,\ \int e^{t-2}\,dt,\ \pi \int \sin(\pi t)\,dt \right\rangle$$

$$\vec{r}(t) = \left\langle 4\ln|t-1|,\ e^{t-2},\ -\cos(\pi t) \right\rangle + \langle c_1, c_2, c_3 \rangle$$

plug $\vec{r}(2) = \langle -5, 3, -2 \rangle$ into the equation to find $\vec{c}$

$\vec{c}$ all of the constants of integration

$$\vec{r}(2) = \langle 4\ln|1|,\ e^0,\ -\cos(2\pi) \rangle + \langle c_1, c_2, c_3 \rangle = \langle -5, 3, -2 \rangle$$

$$\langle 0, 1, -1 \rangle + \langle c_1, c_2, c_3 \rangle = \langle -5, 3, -2 \rangle$$

$$\langle c_1, c_2, c_3 \rangle = \langle -5, 2, -1 \rangle$$

$$\boxed{\vec{r}(t) = \left\langle 4\ln|t-1| - 5,\ e^{t-2} + 2,\ -\cos(\pi t) - 1 \right\rangle}$$

4. (7 points) Find the arc length function $s(t)$ for $\mathbf{r}(t) = \langle 3t+1, 4t-5, 2t\rangle$.

$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du \quad , \quad \vec{r}\,'(t) = \langle 3, 4, 2 \rangle$$

$$= \int_a^t \sqrt{(3)^2 + (4)^2 + (2)^2}\, du$$

$$= \int_a^t \sqrt{29}\, du = \sqrt{29}\, u \Big|_a^t = \underline{\sqrt{29}\,(t-a)}$$

5. (9 points) Find the curvature of $\mathbf{r}(t) = \langle e^{-2t}, \frac{1}{2}t^2, 4\rangle$ at $t = 0$. (Hint: use the cross product version of the curvature.)

$$\vec{v} = \vec{r}\,' = \langle -2e^{-2t}, t, 0 \rangle \qquad |\vec{r}\,'| = \sqrt{(-2e^{-2t})^2 + (t)^2 + 0^2}$$

$$\vec{a} = \vec{r}\,'' = \langle 4e^{-2t}, 1, 0 \rangle \qquad = \sqrt{4e^{-4t} + t^2}$$

$$\vec{r}\,'' \times \vec{r}\,' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4e^{-2t} & 1 & 0 \\ -2e^{-2t} & t & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ t & 0 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 4e^{-2t} & 0 \\ -2e^{-2t} & 0 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 4e^{-2t} & 1 \\ -2e^{-2t} & t \end{vmatrix}\hat{k}$$

$$= 0\hat{\imath} - 0\hat{\jmath} + \left(4te^{-2t} + 2e^{-2t}\right)\hat{k}$$

$$\kappa(t) = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3} = \frac{\sqrt{0^2 + 0^2 + (4te^{-2t} + 2e^{-2t})^2}}{(4e^{-4t} + t^2)^{3/2}}$$

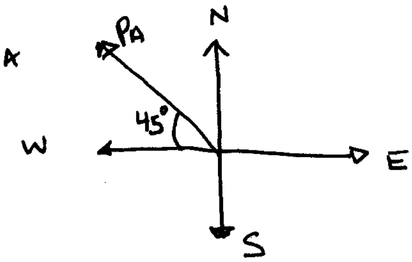
$$= \frac{4te^{-2t} + 2e^{-2t}}{(4e^{-4t} + t^2)^{3/2}}$$

Plug in $t=0$

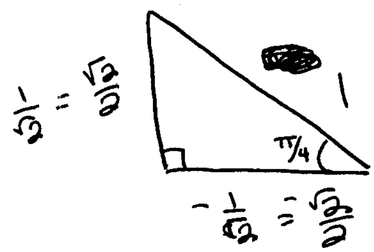
$$\kappa(0) = \frac{0 + 2(1)}{(4(1) + 0)^{3/2}} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

6. (8 points) An airplane flies northwest at a constant altitude at 550 mi/hr relative to the air. There is a southernly crosswind with a magnitude of 40 mi/hr. Find the velocity vector of the airplane relative to the ground. You may leave your answer in terms of radicals.

plane $_A$



Reference triangle for plane



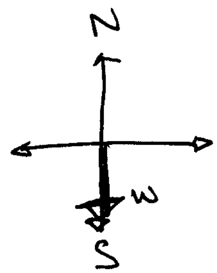
negative since the angle is $\frac{3\pi}{4}$ direction → puts us in Quad II

Direction : $3\pi/4$

Magnitude : 550 mi/hr

Components : $550\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$

wind



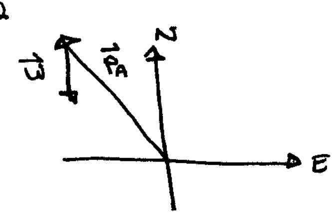
Direction : $\frac{3\pi}{2}$

magnitude : 40

Components : $\langle 0, -40\rangle$

↑ negative since the angle/direction is $3\pi/2$

Plane's velocity relative to the ground is $\vec{P_A} + \vec{w}$



$$\vec{P_g} = 550\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle + \langle 0, -40\rangle$$

$$\boxed{\vec{P_g} = \langle -275\sqrt{2}, 275\sqrt{2} - 40\rangle}$$

7. (6 points) Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. Show that $(\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$ if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. (Hint: start with the left side of the equation and rewrite it.)

NOTE: This is a proof. Do NOT use a vector with specific values.

$$(u+v)\bullet(u+v) = (u+v)\bullet u + (u+v)\bullet v$$
$$= (u\bullet u) + (v\bullet u) + (u\bullet v) + v\bullet v$$

Note that we have
$v\bullet u = u\bullet v$
& $u\bullet v = 0$ since u & v are othogonal

$$= u\bullet u + 2v\bullet u + v\bullet v$$
$$= u\bullet u + v\bullet v$$

Note: $u\bullet u = |u|^2$

$$= |u|^2 + |v|^2 \quad \blacksquare$$

EXTRA. (4 points) Let a and b be scalars and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Prove the following: $(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$. (NOTE: This is a proof. Do not use vectors with specific values.)

$$(au) \times (bv) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ au_1 & au_2 & au_3 \\ bv_1 & bv_2 & bv_3 \end{vmatrix} = \begin{vmatrix} au_2 & au_3 \\ bv_2 & bv_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} au_1 & au_3 \\ bv_1 & bv_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} au_1 & au_2 \\ bv_1 & bv_2 \end{vmatrix} \hat{k}$$

$$= (au_2 \cdot bv_3 - au_3bv_2)\hat{\imath} - (au_1bv_3 - au_3bv_1)\hat{\jmath} + (au_1bv_2 - au_2bv_1)\hat{k}$$

$$\ast = (ab)\left[(u_2v_3 - u_3v_2)\hat{\imath} - (u_1v_3 - u_3v_1)\hat{\jmath} + (u_1v_2 - v_1u_2)\hat{k}\right]$$

$$= (ab)(u \times v)$$

* Can show an intermediate step of factoring

Problem	Max Points	Points
1	6	
2	9	
3	15	
4	7	
5	9	
6	8	
7	6	
Extra	4	
Total	60	