1. (12 points) Consider the surface below.

$$x^2 + \frac{y^2}{4} + z^2 = 1$$

(a) (6 points) Find the xy, xz, and yz traces of the surface.

xy- trace: $x^2 + \frac{y^2}{4} = 1$

xz- trace: $x^2 + z^2 = 1$

yz- trace: $\frac{y^2}{4} + z^2 = 1$

(b) (1 point) What type of surface is the function?

Ellipsoid

(c) (5 points) Find the equation of the tangent plane to the surface at the point $(0, 2, 0)$.

Gradient $\nabla f = \langle 2x, \frac{1}{2}y, 2z \rangle$

$\nabla f(0,2,0) = \langle 0, 1, 0 \rangle$

The tangent plane is

$0 + 1(y-2) + 0 = 0$

$\boxed{y = 2}$

2. (9 points) Find the partial derivatives f_x, f_y, f_z for the function below.
DO NOT SIMPLIFY YOUR ANSWER.

$$f(x, y, z) = \frac{x^2 + y}{xyz + z^2} = (x^2+y)(xyz+z^2)^{-1}$$

$$f_x = \frac{(2x)(xyz+z^2) - (x^2+y)(yz)}{(xyz+z^2)^2} = (2x)(xyz+z^2)^{-1} - (x^2+y)(xyz+z^2)^{-2}(yz)$$

$$f_y = \frac{(1)(xyz+z^2) - (x^2+y)(xz)}{(xyz+z^2)^2} = (1)(xyz+z^2)^{-1} - (x^2+y)(xyz+z^2)^{-2}(xz)$$

$$f_z = \frac{-(xy+2z)(x^2+y)}{(xyz+z^2)^2} = -(x^2+y)(xyz+z^2)^{-2}(xy+2z)$$

Quotient Rule   product Rule

3. (5 points) For the following set of three planes, determine if the planes are parallel, orthogonal, or none of the above.

$$Q: -2x - 2y + 4z = 2 \qquad R: x + y + z = 1 \qquad S: -2x - 2y - 2z = 15$$

normal vectors: $\langle -2, -2, 4 \rangle$ $\quad\langle 1, 1, 1 \rangle$ $\quad\langle -2, -2, -2 \rangle$

R & S : parallel planes by inspection of their normal vectors (the vectors are scalar multiples of each other)

R & Q : $\langle -2, -2, 4 \rangle \cdot \langle 1, 1, 1 \rangle = -2 + -2 + 4 = 0$
Therefore, the planes are orthogonal since their normal vectors are orthogonal.

S & Q : These planes are orthogonal as well since S is parallel to R and R is orthogonal to Q

4. (12 points) Consider the function below.

$$f(x,y) = \frac{2y}{x^2+y}$$

(a) (2 points) Compute the limit as $(x,y) \to (0,0)$ along the y-axis.

$$\lim_{(x,y)\to(0,0)} \frac{2y}{0+y} = \lim_{(x,y)\to(0,0)} 2 = \boxed{2}$$

(b) (2 points) Compute the limit as $(x,y) \to (0,0)$ along the line $y = mx$, $m \neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{2mx}{x^2+mx} = \lim_{(x,y)\to(0,0)} \frac{2m}{x+m} = \frac{2m}{m} = \boxed{2}$$

(c) (2 points) Compute the limit as $(x,y) \to (0,0)$ along the parabola $y = x^2$.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{2x^2} = \lim_{(x,y)\to(0,0)} 1 = \boxed{1}$$

(d) (3 point) What can you conclude about the limit at $(0,0)$? Explain why.

The limit does not exist because the limit has different values along two paths.

(e) (3 points) What can you conclude about the continuity of the function below at $(0,0)$? Explain your reasoning.

$$g(x,y) = \begin{cases} \frac{2y}{x^2+y} & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0) \end{cases}$$

The function $g(x,y)$ is discontinuous at $(0,0)$ because the limit does not exist.

5. (12 points) The area of a right triangle with base b and height h is $A = \frac{1}{2}bh$.

(a) (4 points) Assume that b and h change according to the function $b = g(t)$ and $h = f(t)$. Find $\frac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{\partial A}{\partial b} \cdot \frac{db}{dt} + \frac{\partial A}{\partial h} \cdot \frac{dh}{dt}$$

$$= \frac{1}{2}h \cdot g'(t) + \frac{1}{2}b\, f'(t)$$

(b) (6 points) Suppose that the sides of the triangle oscillate in length according to $b = 1 + \cos^2(t)$ and $h = 1 + \sin^2(t)$. Find $\frac{dA}{dt}$.

$$\frac{db}{dt} = -2\cos t \sin t \qquad \frac{dh}{dt} = 2\sin t \cos t$$

$$\frac{dA}{dt} = \frac{1}{2}h(-2\cos t \sin t) + \frac{1}{2}b(2\sin t \cos t)$$

$$= -[1+\sin^2 t](\sin t \cos t) + (1+\cos^2 t)(\sin t \cos t)$$

$$= (\sin t \cos t)(\cos^2 t - \sin^2 t)$$

(c) (2 points) Is the area of the triangle in part (b) increasing or decreasing at $t = \frac{\pi}{2}$?

$$\frac{dA}{dt} = \left(\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right)\left(\cos^2\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$$

$$= (1 \cdot 0)(0 - 1)$$

$$= \boxed{0}$$

The area is not changing

6. (12 points) Use the Second Derivative Test to classify the critical point(s) of the function below.

   Note: You do NOT have to compute function values.

$$f(x, y) = x^2 + y^2 + x^2y + 4$$

$f_x = 2x + 2xy \longrightarrow 2x(1+y) = 0$

$f_y = 2y + x^2$

$x = 0 \quad y = -1$

plug into f_y

$x = 0$: $2y + 0 = 0$, $y = 0$ $\boxed{(0,0)}$

$y = -1$: $-2 + x^2 = 0$, $x = \pm\sqrt{2}$ $\boxed{(\pm\sqrt{2}, -1)}$

$f_{xx} = 2 + 2y$

$f_{yy} = 2$

$f_{yx} = 2x$

Analysis of each point

$(0,0)$: $D(0,0) = (2+0)(2) - (2\cdot 0)^2 = 4 > 0$

$f_{yy} > 0$

$\boxed{\text{rel. min}}$

$(\sqrt{2}, -1)$: $D(\sqrt{2}, -1) = (2 + 2(-1))(2) - (2\sqrt{2})^2$

$= 0 - 8 < 0$

$\boxed{\text{Saddle point}}$

$(-\sqrt{2}, -1)$: $D(\sqrt{2}, -1) = (2 + 2(-1))(2) - (-2\sqrt{2})^2$

$= 0 - 8 < 0$

$\boxed{\text{Saddle point}}$

7. (8 points) Using Lagrange multipliers, find the point at which the maximum occurs for the given function and constraint.

$$f(x,y) = 12xy - x^2 - 3y^2 \qquad \text{subject to} \quad x + y = 16$$

$f_x = \lambda g_x$ $\quad$ $f_y = \lambda g_y$ $\quad$ $g(x,y) = 0$

$12y - 2x = \lambda$ $\quad$ $12x - 6y = \lambda$ $\quad$ $x+y = 16$

$12y - 2x = 12x - 6y$

$18y = 14x$

$y = \frac{14}{18}x$

$y = \frac{7}{9}x \xrightarrow{\text{plug into constraint}} x + \frac{7}{9}x = 16$

$\frac{16}{9}x = 16$

$\boxed{x = 9}$

$y = \frac{7}{9}(9)$

$\boxed{y = 7}$

$\boxed{\text{The point is } (9,7)}$

EXTRA. (4 points) Find the limit (it does exist).

$$\lim_{(x,y)\to(4,0)} x^2 y \ln(xy)$$

$$= \left(\lim_{(x,y)\to(4,0)} x\right)\left(\lim_{(x,y)\to(4,0)} xy \cdot \ln(xy)\right)$$

$$= 4 \cdot (0 \cdot -\infty) \leftarrow \text{indeterminant}$$

use $xy = u$

$$= 4 \cdot \lim_{u\to 0} u \cdot \ln(u)$$

$$= 4 \lim_{u\to 0} \frac{\ln(u)}{\frac{1}{u}}$$

$$= 4 \ \frac{-\infty}{\infty} \leftarrow \text{also indeterminant}$$

$$= 4 \lim_{u\to 0} \frac{\frac{1}{u}}{-\frac{1}{u^2}} = 4 \cdot \lim_{u\to 0} -u$$

$$= 4 \cdot 0 = \boxed{0}$$

Problem	Max Points	Points
1	12	
2	9	
3	5	
4	12	
5	12	
6	12	
7	8	
Extra	4	
Total	70	