1. (12 points) Consider the two line integrals.

A) $\int_C \langle y^2, 2xy\rangle \bullet d\mathbf{r}$ B) $\int_C \langle x^2, 2xy\rangle \bullet d\mathbf{r}$

(a) (5 points) For which integral does the Fundamental Theorem of Line Integrals apply? Explain.

The Fundamental Thm for line integrals applies only to conservative vector fields. $\vec{F}$ is conservative if the mixed partials are equal.

[A] $\frac{\partial}{\partial y} y^2 \overset{?}{=} \frac{\partial}{\partial x} 2xy$

$2y = 2y$ ✓

[B] $\frac{\partial}{\partial y} x^2 \overset{?}{=} \frac{\partial}{\partial x} 2xy$

$0 \neq 2y$

FTLI applies only to integral A.

(b) (7 points) If the end points of a curve C are $P(2,4)$ and $Q(1,0)$, calculate the value of the appropriate line integral.

Need to find the potential function.

$\vec{F} = \langle \varphi_x, \varphi_y \rangle = \langle y^2, 2xy \rangle$

$\varphi = \int y^2 dx = xy^2 + c(y) \rightarrow \frac{\partial \varphi}{\partial y} = 2xy + c'(y) = \varphi_y$ from $\vec{F}$

$2xy + c'(y) = 2xy$

$c'(y) = 0$ let $c(y) = 0$

so $\varphi(x,y) = xy^2$

use FTLI

$\int_C \langle y^2, 2xy \rangle \cdot d\vec{r} = \varphi(Q) - \varphi(P)$

$= (1)(0)^2 - (2)(4)^2$

$= \boxed{-32}$

2. (9 points) Find the potential function for the gradient field

$$\mathbf{F} = \langle yz, xz, xy + 2z \rangle$$

$\varphi_x \quad \varphi_y \quad \varphi_z$

Integrate

$\int \varphi_x \, dx = \int yz \, dx$

$\varphi = xyz + c^*(y) + c(z)$

$\frac{\partial \varphi}{\partial y} = xz + c'(y) = xz$

$c'(y) = 0$ let $c(y) = 0$ or $c(y) = c_1$

$\frac{\partial \varphi}{\partial z} = xy + c'(z) = xy + 2z$

$c'(z) = 2z$

$\int c'(z)\,dz = \int 2z \, dz$

$c^*(z) = z^2 + c_2$

$\boxed{\varphi = xyz + z^2 + C}$

3. (9 points) Parameterize the surface and give the bounds on the parameters.
Cylinder $y^2 + z^2 = 16$ between $x = 0$ and $x = 5$.

→ Given a cylinder → use cylindrical coordinates

$y = 4\cos\theta \quad z = 4\sin\theta \quad x = v \quad$ (or could say $x = x$)

$= 4\cos u \quad = 4\sin u$

parameterization: $\vec{r}(\theta, v) = \langle v, 4\cos\theta, 4\sin\theta \rangle$

Bounds: $0 \le \theta \le 2\pi$

$0 \le v \le 5$

4. (10 points) Answer each

(a) (5 points) Consider the vector field $\mathbf{F} = \langle xy, 3y^2\rangle$ on the parameterized curve $C : \mathbf{r}(t) = \langle 11t^4, t^3\rangle$ for $0 \le t \le 1$. Set up the line integral for finding circulation.

$$\int_C \vec{F} \cdot \vec{r}\,' \, dt \qquad \vec{r}\,' = \langle 44t^3, 3t^2 \rangle$$

$$\vec{F} = \langle (11t^4)(t^3), (3)(t^3)^2 \rangle = \langle 11t^7, 3t^6 \rangle$$

$$\int_0^1 \langle 11t^7, 3t^6 \rangle \cdot \langle 44t^3, 3t^2 \rangle \, dt$$

(b) (5 points) Consider $f(x, y) = xy$ on the curve $C : \mathbf{r}(t) = \langle t^2, 2t\rangle$ for $0 \le t \le 1$. Calculate the line integral.

$$\int_C f \cdot \|\vec{r}\,'\| \, dt \qquad \|\vec{r}\,'\| = \sqrt{(2t)^2 + (2)^2} = \sqrt{4t^2 + 4} = 2\sqrt{t^2+1}$$

$$f = t^2(2t) = 2t^3$$

accepted Answer →

$$\boxed{\int_0^1 2t^3 \cdot 2\sqrt{t^2+1} \, dt}$$

$$= 2\int_0^1 2t^3 \sqrt{t^2+1} \, dt \qquad u = t^2+1 \quad du = 2t \qquad t^2 = u-1$$

$$= 2\int_0^1 (u-1)\sqrt{u} \, du$$

$$= 2\int_0^1 u^{3/2} - u^{1/2} \, du$$

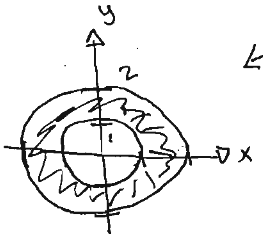
$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1$$

$$= \frac{4}{5}(t^2+1)^{5/2} - \frac{4}{3}(t^2+1)^{3/2}\Big|_0^1$$

$$\boxed{\left[\frac{4}{5}(2)^{5/2} - \frac{4}{3}(2)^{3/2}\right] - \left[\frac{4}{5} - \frac{4}{3}\right]}$$

5. (14 points) Consider the vector field $\mathbf{F} = \langle \cos y, x^2 \sin y \rangle$ on curve C which is the boundary between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(a) (7 points) Using Green's Theorem, set up the integral for circulation.



← working on a circle → use polar

$x = r\cos\theta$

$y = r\sin\theta$

$\iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dA$

$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x\sin y + \sin y$

$= 2r\cos\theta \sin(r\sin\theta) + \sin(r\sin\theta)$

$$\boxed{\int_0^{2\pi}\int_1^2 (2r\cos\theta + 1)\sin(r\sin\theta)\cdot r\,dr\,d\theta}$$

(b) (7 points) Using Green's Theorem, set up the integral for flux.

divergence : $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + x^2\cos y$

$= (r\cos\theta)^2 \cos(r\sin\theta)$

$\iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \, dA$

$$\boxed{= \int_0^{2\pi}\int_1^2 (r\cos\theta)^2 \cos(r\sin\theta)\, r\,dr\,d\theta}$$

6. (8 points) Use the Divergence Theorem to set up an integral to calculate the flux of $\mathbf{F} = \langle 2x, 2y^3, 2z^3 \rangle$ where surface S is the hemisphere $x^2 + y^2 + z^2 = 9$ where $z \geq 0$.

You may use either formula, but state which formula you want to use.

One parameterization of the curve is:

$$u = \phi \qquad v = \theta$$

$$\mathbf{r}(u, v) = \langle 3 \sin u \cos v, 3 \sin u \sin v, 3 \cos u \rangle$$

$$\iiint_V \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot n \, dS$$

Volume: $\nabla \cdot \vec{F} = 2 + 6y^2 + 6z^2$

Bounds for spherical coordinates: $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$, $0 \leq \rho \leq 3$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{3} \left[2 + 6(3 \sin\phi \sin\theta)^2 + 6(3\cos\phi)^2 \right] \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Surface:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos u \cos v & 3 \sin u \cos v & -3\sin u \\ -3 \sin u \sin v & 3 \sin u \cos v & 0 \end{vmatrix}$$

$$\vec{n} = \vec{t}_u \times \vec{t}_v = 9 \langle \sin^2 u \cos v, -\sin^2 u \sin v, \cos u \sin u \rangle$$

$$\vec{F} = \langle 2 \cdot 3 \sin u \cos v, 2(3 \sin u \sin v)^3, 2(3\cos u)^3 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \int_0^{2\pi} \int_0^{\pi/2} \langle 6 \sin u \cos v, 54 \sin^3 u \sin^3 v, 54 \cos^3 u \rangle \cdot \vec{n} \, d\phi \, d\theta$$

7. (8 points) Use Stokes' Theorem to set up an integral to calculate the circulation of $\mathbf{F} = \langle y^2, x, z^2\rangle$ on the surface S that is part of the plane $2x + y + z = 2$ that lies in the first octant and it oriented upward.

You may use either formula, but state which formula you want to use.

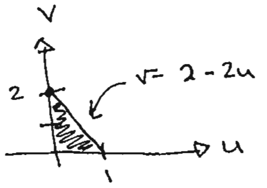
One parameterization of the curve is:

$$x = u \qquad y = v \qquad z = 2 - 2u - v$$
$$\mathbf{r}(u, v) = \langle u, v, 2 - 2u - v\rangle$$

Surface Integral: $\iint_S (\nabla \times F)\cdot \vec{n}\, dS$

curl : $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (1-2y)\hat{k}$

$= \langle 0, 0, 1-2y\rangle$



$$\int_0^1\int_0^{2-2u} \langle 0,0,1-2v\rangle \cdot \langle 2,1,1\rangle \, dv\, du$$

$$= \boxed{\int_0^1\int_0^{2-2u} 1 - 2v \, dv\, du}$$

$t_u = \langle 1, 0, -2\rangle$

$t_v = \langle 0, 1, -1\rangle$

$\vec{n} = (2)\hat{\imath} - (-1)\hat{\jmath} + (1)\hat{k}$

$= \langle 2, 1, 1\rangle$ → z is positive

Line Integral $\int_C \vec{F}\cdot\vec{r}'\, dt$ Need to parameterize the triangle → difficult b/c need 3 integrals

EXTRA. (4 points) For vector field $\mathbf{F} = \langle f, g, h \rangle$ (f, g, h have continuous second partial derivative), prove

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

In words, the divergence of the curl is 0.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) \hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}$$

So

$$\nabla \cdot (\nabla \times F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$$= \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 h}{\partial y \partial x} + \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}$$

$$= \underbrace{\frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x}}_{0} + \underbrace{\frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 g}{\partial x \partial z}}_{0} + \underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{0}$$

$$= 0$$

since $\vec{F}$ is conservative, the mixed partials of $\vec{F}$ are equal.

Problem	Max Points	Points
1	12	
2	9	
3	9	
4	10	
5	14	
6	8	
7	8	
Extra	4	
Total		