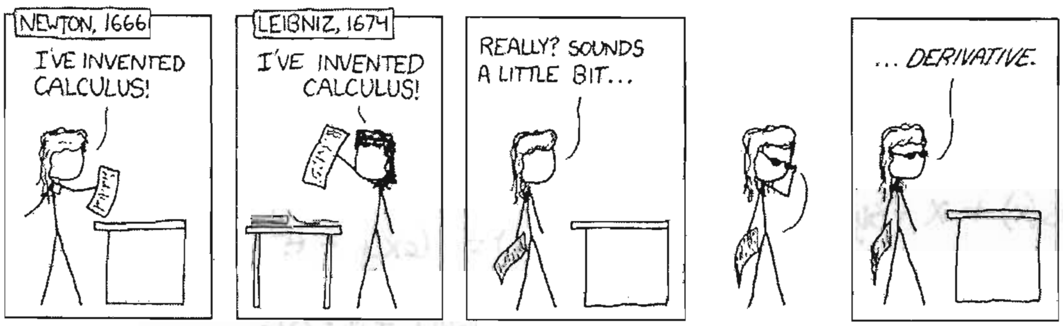
MATH 420 – Differential Equations
American River College
Kristin Lui

Name: Key

Student ID: ____________________

EXAM 1

- The exam is closed book, notes and neighbor. No calculators.
- SHOW ALL WORK!!!
- Good luck!



Problem	1	2	3	4	5	6	Bonus	Total
Score								
Possible	16	18	18	14	20	14	10	100

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1. (16 points) For the autonomous differential equation, find and classify the critical point(s) as unstable, semi-stable, or asymptotically stable. Draw the phase portrait.

$$\frac{dy}{dx} = y^2(4 - y^2)$$

Critical Points

$y' = 0 \to y^2(4-y^2) = 0$

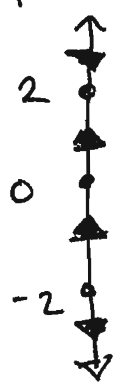
$y^2 = 0 \quad 4 - y^2 = 0$

$\boxed{y = 0 \quad y = \pm 2}$

Stability Analysis

	y	dy/dx	sign
	3	$+(-)$	$-$
$y=2$	1	$(+)(+)$	$+$
$y=0$	-1	$(+)(+)$	$+$
$y=-2$	-3	$(+)(-)$	$-$

phase portrait



$y = 2$ (asymptotically) stable

$y = 0$ semi-stable

$y = -2$ unstable

2. (18 points) Solve the exact differential equation. Find a solution satisfying the initial condition $\underline{y(0) = 0}$.

$$(\cos(x) + x + 3y^2)dx + (6xy + e^y)dy = 0$$

$M = \cos(x) + x + 3y^2$

$\int \cos(x) + x + 3y^2\, dx$

$= \sin(x) + \frac{1}{2}x^2 + 3xy^2 + g(y)$

$N = 6xy + e^y$

$\int 6xy + e^y\, dy$

$= 3xy^2 + e^y + g(x)$

$$h(x,y) = \sin(x) + 3xy^2 + \tfrac{1}{2}x^2 + e^y = C$$

Plug in IC $y(0) = 0$

$\sin(0) + 3(0)(0) + \frac{1}{2}(0) + e^0 = c$

$\boxed{1 = c}$

$$\boxed{\begin{gathered}\sin(x) + 3xy^2 + \tfrac{1}{2}x^2 + e^y = 1 \\ \text{OR} \\ \sin(x) + 3xy^2 + \tfrac{1}{2}x^2 + e^y - 1 = 0\end{gathered}}$$

← Note placement of constant $c = 1$

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3. (18 points) Solve the given Initial Value Problem:

$$(2x+3)y' = y + (2x+3)^{1/2} \qquad y(-1) = 4$$

Standard form

$y' - (2x+3)^{-1} y = (2x+3)^{-1/2}$

integrating factor

$e^{-\int \frac{1}{2x+3} dx} = e^{-\frac{1}{2}\ln|2x+3|} = (2x+3)^{-1/2}$

$(2x+3)^{-1/2} y' - (2x+3)^{-3/2} y = (2x+3)^{-1}$

$\int \frac{d}{dx}\left[(2x+3)^{-1/2} \cdot y\right] dx = \int (2x+3)^{-1} dx$

$\boxed{(2x+3)^{-1/2} \cdot y = \frac{1}{2}\ln|2x+3| + C}$

Solve for c

$(2(-1)+3)^{-1/2} \cdot 4 = \frac{1}{2}\ln|2(-1)+3| + C$

$(1) \cdot 4 = \frac{1}{2}\ln|1| + C$

$\boxed{4 = C}$

$\boxed{(2x+3)^{-1/2} \cdot y = \frac{1}{2}\ln|2x+3| + 4}$

4. (14 points) Solve the homogeneous differential equation

$$y' = \frac{2-x-y}{x+y}$$

$\frac{dy}{dx} = \frac{2-(x+y)}{x+y}$

$u = x+y$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

substitution

$\frac{du}{dx} - 1 = \frac{2-u}{u}$

$\frac{du}{dx} = \frac{2-u}{u} + \frac{u}{u}$

$\frac{du}{dx} = \frac{2}{u}$

$u\,du = 2dx$

$\int u\,du = \int 2dx$

$\frac{1}{2}u^2 = 2x + C$

$\boxed{\frac{1}{2}(x+y)^2 = 2x + C}$

5. (20 points) A full 400 gallon water tank contains 150 lbs of salt. Water containing 1 lb/gal of salt flows into the tank at a rate of 4 gal/min. The well mixed solution flows out at the same rate.
   a. What is the differential equation and initial value for the system?
   b. Solve the initial value problem.
   c. How much salt is in the tank after a long, long time? Provide mathematical justification.

A) Let $A(t)$ = amount of salt at any time

$$\frac{dA}{dt} = 4 - \frac{A}{400}\cdot\frac{4}{1} = \boxed{4 - \frac{A}{100} = \frac{dA}{dt}}$$

$$\boxed{A(0) = 150}$$

B) $\frac{dA}{dt} + \frac{A}{100} = 4$ int. factor: $e^{\frac{1}{100}\int dt} = e^{\frac{1}{100}\cdot t}$

$$e^{\frac{t}{100}}\left[\frac{dA}{dt} + \frac{A}{100}\right] = e^{\frac{t}{100}}\cdot 4$$

$$\int \frac{d}{dt}\left[e^{t/100}\cdot A\right]dt = 4\int e^{t/100}\,dt$$

$$e^{t/100}\cdot A = 4\cdot 100\,e^{t/100} + C$$

$$A = 400 + Ce^{-t/100}$$

plug in IC

$$150 = 400 + C$$

$$-250 = C$$

$$\boxed{A = 400 - 250e^{-t/100}}$$

c) As $t \to \infty$, the $e^{-t/100}$ term goes to 0 and $A = 400$

$$\lim_{t\to\infty} A(t) = 400 - 0 = 400 \text{ lbs of salt}$$

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6. (14 points) Solve the differential equation with the appropriate substitution:

$$\frac{dy}{dx} = (x-y)^2 - 2(x-y) - 2$$

$u = x - y$

$\frac{du}{dx} = 1 - \frac{dy}{dx} \to \frac{dy}{dx} = 1 - \frac{du}{dx}$

$1 - \frac{du}{dx} = u^2 - 2u - 2$

$\frac{du}{dx} = -u^2 + 2u + 1$

$\frac{du}{dx} = -(u-1)^2$

$\int \frac{du}{(u-1)^2} = \int - dx$

$\to$

$-(u-1)^{-1} = -x + C$

$\boxed{-(x-y-1)^{-1} = -x + C}$

BONUS: Graph the solution curves for problem 1. Be sure to show concavity! Tip: $\sqrt{2} \approx 1.41$

$\frac{d^2y}{dx^2} = 2y \cdot \frac{dy}{dx} + (-2y)y^2(dy/dx) = \frac{dy}{dx}(2y - 2y^3) = (y^2)(4-y^2)(2y(1-y^2))$

$= 2y^3(4-y^2)(1-y^2)$

inflection points:

$\frac{d^2y}{dx^2} = 0 \to y = 0 \quad y = \pm 1$
$y = \pm 2$

Stability Analysis

y	$d^2y/dx^2 = y''$	Sign	
3	$(+)(-)(-)$	+	∪
$\sqrt{2}$	$(+)(+)(-)$	−	∩
$1/2$	$(+)(+)(+)$	+	∪
$-1/2$	$(-)(+)(+)$	−	∩
$-\sqrt{2}$	$(-)(+)(-)$	+	∪
-3	$(-)(-)(-)$	−	∩

Sol'n Curves