MATH 420 – Differential Equations
Kristin Lui

1. (14 points) Consider the differential equation below.

$$x''' + 6x'' + 9x' = g(x)$$

   a. (6 points) Find the complementary solution to the homogeneous differential equation.

auxillary eqn : $m^3 + 6m^2 + 9m = 0$

$m(m^2 + 6m + 9) = 0$

$m(m+3)^2 = 0$

$m = 0 \quad m = -3$ w/ multiplicity 2

$$\boxed{y_c = c_1 + c_2 e^{-3x} + c_2 x e^{-3x}}$$

   b. (8 points) For the DE described above, let $g(x)$ be defined below. For each different $g(x)$, guess the form of the particular solution and give the proper annihilator function. Use the space below for work before writing your answer.

i. $g(x) = 4x^3$

Guess form of y_p: $Ax^3 + Bx^2 + Cx + D$ modify to: $Ax^4 + Bx^3 + Cx^2 + Dx$

Annihilator Function: D^4

ii. $g(x) = 5xe^{-3x}$

Guess form of y_p: $(Ax+B)e^{-3x}$ modify to $x^2(Ax+B)e^{-3x}$

Annihilator Function: $(D+3)^2$

iii. $g(x) = xe^{-3x}\sin(4x)$

Guess form of y_p: $e^{-3x}\left[(Ax+B)\sin(4x) + (Cx+D)\cos(4x)\right]$

Annihilator Function: $[D^2 + 6D + 25]^2$

Space below is for work if needed.

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2. (16 points) Solve the differential equation using the **superposition approach** for undetermined coefficients. Little or no credit will be given if the annihilator approached is used.

$$y'' - y' - 2y = 6x + 6e^{-x}$$

Solve for y_c

auxillary eqn

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2 \quad m = -1$

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{2x}}$$

Guess: $y_p = Ax + B + Cxe^{-x}$

$y_p' = A + Ce^{-x} - Cxe^{-x}$

$y_p'' = -Ce^{-x} - Ce^{-x} + Cxe^{-x} = -2Ce^{-x} + Cxe^{-x}$

$$-2Ce^{-x} + \underbrace{Cxe^{-x}}_{=0} - A - Ce^{-x} + \underbrace{Cxe^{-x}}_{=0} - 2Ax - 2B - \underbrace{2Cxe^{-x}}_{=0} = 6x + 6e^{-x}$$

$$-3Ce^{-x} - 2Ax - A - 2B = 6x + 6e^{-x}$$

e^{-x} terms	x terms	constants
$-3C = 6$	$-2A = 6$	$-A - 2B = 0$
$\boxed{C = -2}$	$\boxed{A = -3}$	$3 = 2B$
		$\boxed{3/2 = B}$

$$\boxed{y_p = -3x + 3/2 - 2xe^{-x}}$$

$$\boxed{y = y_c + y_p}$$

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3. (16 points) Solve the system of differential equations.

$$\begin{cases} x' = x - 2y \\ y' = 5x - y \end{cases}$$

$$\to \begin{cases} Dx - x + 2y = 0 \\ Dy - 5x + y = 0 \end{cases} \to \begin{cases} (D-1)x + 2y = 0 \\ -5x + (D+1)y = 0 \end{cases}$$

Eliminate x, multiply 1st eqn by 5
" 2nd eqn by $(D-1)$:

$$\begin{cases} 5(D-1)x + 10y = 0 \\ -5(D-1)x + (D^2-1)y = 0 \end{cases}$$

Eqns Combined:

$$10y + (D^2-1)y = 0$$

$$(D^2+9)y = 0$$

$$m^2 + 9 = 0 \to m = \pm 3i$$

$$\boxed{y_c = C_1 \sin(3t) + C_2 \cos(3t)}$$

Find $x_c(t)$ by using the eqn from the system

$$y' = 5x - y$$

$$y' = 3C_1 \cos(3t) - 3C_2 \sin(3t)$$

$$x = \frac{1}{5}(y' + y)$$

$$x = \frac{1}{5}\left[3C_1 \cos(3t) - 3C_2 \sin(3t) + C_1 \sin(3t) + C_2 \cos(3t)\right]$$

$$\boxed{x = \frac{1}{5}\left[(3C_1 + C_2)\cos(3t) + (C_1 - 3C_2)\sin(3t)\right]}$$

Sol'n set :

$$\begin{cases} y = C_1 \sin(3t) + C_2 \cos(3t) \\ x = \left(\frac{3C_1 + C_2}{5}\right)\cos(3t) + \left(\frac{C_1 - 3C_2}{5}\right)\sin(3t) \end{cases}$$

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4. (20 points) A spring attached to the ceiling is stretched by 6 inches by a 2lb weight. The spring is in a medium that imparts a damping force of 1lb·sec/ft. The mass is pulled down 3 inches and imparted with an upward velocity of 3 ft/s.
   a. (7 points) What is the DE of the system and the initial conditions?
   b. (11 points) Find the equation of motion that satisfies the initial conditions.

a) $W = 2 \rightarrow m = \frac{2}{32} = \frac{1}{16}$

$s = \frac{1}{2}\text{ft} \quad k = \frac{2}{1/2} = 4$

$\beta = 1$

$$\boxed{\begin{array}{l} \frac{1}{16}x'' + x' + 4x = 0 \\ x(0) = 3\text{in} = 1/4\text{ ft} \\ x'(0) = -3 \end{array}}$$

b) $x'' + 16x' + 64x = 0$

$m^2 + 16m + 64 = 0$

$(m+8)^2 = 0$

$m = -8$ w/ multiplicity 2

$$\boxed{x_c = c_1 e^{-8t} + c_2 t e^{-8t}}$$

Solve the IVP

$x_c(t) = c_1 e^{-8t} + c_2 t e^{-8t}$

$x' = -8c_1 e^{-8t} + c_2 e^{-8t} - 8c_2 t e^{-8t}$

$x(0) = \frac{1}{4} = c_1 \cdot 1 + 0$

$\boxed{\frac{1}{4} = c_1}$

$x'(0) = -8c_1 \cdot 1 + c_2 \cdot 1 + 0 = -3$

$-2 + c_2 = -3$

$\boxed{c_2 = -1}$

$$\boxed{x_c(t) = \frac{1}{4}e^{-8t} - te^{-8t}}$$

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5. (16 points) Use **variation of parameters** to solve the non-homogeneous differential equation.

$$x^2y'' + 8xy' + 6y = x^{-1}$$

$$y'' + \frac{8}{x}y' + \frac{6}{x^2} = x^{-3} \qquad \boxed{f(x) = x^{-3}}$$

Solve Cauchy - Euler, hom. DE

$x^2y'' + 8xy' + 6y = 0$

$a = 1 \quad b = 8 \quad c = 6$

$m^2 + 7m + 6 = 0$

$(m+6)(m+1) = 0$

$m = -6 \quad m = -1$

$$\boxed{y_c = C_1 x^{-1} + C_2 x^{-6}}$$

$y_1 = x^{-1} \qquad y_2 = x^{-6}$

$$W(y_1, y_2) = \begin{vmatrix} x^{-1} & x^{-6} \\ -x^{-2} & -6x^{-7} \end{vmatrix} = -6x^{-8} + x^{-8} = -5x^{-8}$$

$$y = u_1 y_1 + u_2 y_2$$

$$u_1 = -\int \frac{x^{-6} \cdot x^{-3}}{-5x^{-8}}\,dx = \frac{1}{5}\int \frac{x^8}{x^9}\,dx = \frac{1}{5}\int \frac{1}{x}\,dx = \frac{1}{5}\ln|x| + C_1$$

$$u_2 = \int \frac{x^{-1}x^{-3}}{-5x^{-8}}\,dx = -\frac{1}{5}\int \frac{x^8}{x^4}\,dx = -\frac{1}{5}\int x^4\,dx = -\frac{1}{25}x^5 + C_2$$

$$y = \frac{1}{5}x^{-1}\cdot\ln|x| + C_1x^{-1} - \underbrace{\frac{1}{25}x^{-1}}_{\text{multiple of } y_1} + C_2\cancel{x}^{-6}$$

$$y = \underbrace{\frac{1}{5}x^{-1}\cdot\ln|x|}_{y_p} + \underbrace{C_1x^{-1} + C_2x^{-6}}_{y_c}$$

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6. (18 points) Use the **reduction of order method** to find the second and particular solutions of the DE.

$$x^2y'' + 8xy' + 6y = x^2 \qquad\qquad y_1(x) = x^{-1}$$

$$y = y_c + y_p = u \cdot y_1$$

$$y = x^{-1} \cdot u$$

$$y' = x^{-1} \cdot u' - x^{-2} \cdot u$$

$$y'' = x^{-1} \cdot u'' - 2x^{-2}u' + 2x^{-3} \cdot u$$

$$x^2y'' + 8xy' + 6y = x^2$$

$$xu'' - 2u' + \underbrace{2x^{-1}u}_{=0} + 8u' - \underbrace{8x^{-1}u}_{=0} + \underbrace{6x^{-1} \cdot u}_{=0} = x^2$$

$$xu'' + 6u' = x^2$$

$$u'' + \frac{6}{x}u' = x$$

Reduce the order:

$w' = u''$

$w = u'$

$$w' + \frac{6}{x}w = x$$

int. factor: $e^{6\int \frac{1}{x}dx} = e^{\ln|x^6|} = x^6$

$$x^6 w' + \frac{6}{x}x^6 w = x^6 \cdot x$$

$$\int \frac{d}{dx}\left[x^6 \cdot w\right]dx = \int x^7\,dx$$

$$x^6 \cdot w = \frac{1}{8}x^8 + C_1$$

$$w = \frac{1}{8}x^2 + C_1x^{-6}$$

$$\int u'\,dx = \int \frac{1}{8}x^2 + C_1x^{-6}\,dx$$

$$u = \frac{1}{24}x^3 + C_1x^{-5} + C_2$$

$$y = u \cdot x^{-1}$$

$$\boxed{y = \underbrace{\frac{1}{24}x^2}_{y_p} + \underbrace{C_1x^{-6}}_{y_2} + \underbrace{C_2x^{-1}}_{y_1}}$$

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Bonus:

a. (4 points) What is the particular solution for the differential equation below? (Hint: this DE is related to problems 5 and 6). **Explain** how you arrived at your answer.

$$x^2y'' + 8xy' + 6y = x^2 + x^{-1}$$

Since the DE is the same for all three problems, we can linearly combine the particular solutions by the super-position principle for non-homogenous DE's:

$$y_p = \frac{1}{24}x^2 + \frac{1}{5}x^{-1}\ln|x|$$

b. (6 points) State the intervals where solutions may exist for the differential equation. State the longest interval on which the DE is certain to have a unique solution for the initial conditions given.

$$(t-1)y'' - 3ty' + 4y = \sin(t) \qquad y(-2) = 2, \;\; y'(-2) = 1$$

$-3t$ cont $\forall t \in \mathbb{R}$

4 cont $\forall t \in \mathbb{R}$

$\sin(t)$ cont $\forall t \in \mathbb{R}$

$t-1$ cont $\forall t \in \mathbb{R}$

but $t-1 \neq 0$

$\boxed{t \neq 1}$

Intervals where sol'ns exist:

$(-\infty, 1) \cup (1, \infty)$

Interval where the unique solution to the IVP exists

$(-\infty, 1)$

since $t = -2$ for the initial conditions.