MATH 420 – Differential Equations
Kristin Lui

1. (15 points) Using the solution about the ordinary point $x = 0$, find the recurrence relation for the DE $$y'' + xy = 0$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} C_n (n) x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$k = n-2$ $\quad$ $k = n+1$

$n = k+2$ $\quad$ $n = k-1$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$$C_2(2)(1) + \sum_{k=1}^{\infty} C_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$$2C_2 + \sum_{k=1}^{\infty} \left[C_{k+2}(k+2)(k+1) + C_{k-1}\right] x^k = 0$$

reccurrence relation $\rightarrow$ $$C_{k+2} = \frac{-C_{k-1}}{(k+2)(k+1)}$$

2. (25 points) Use the Laplace Transform Table for the problems below:

   a. (12 points) Using transforms 18 and 19 in the Laplace transform table, evaluate the inverse transform below.

$$\mathcal{L}^{-1}\left\{\frac{2s-2}{s^2-4s+8}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s-2}{s^2-4s+4+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s-2}{(s-2)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s-4}{(s-2)^2+4} + \frac{4-2}{(s-2)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= \boxed{2\cdot e^{2t}\cos(2t) + e^{2t}\sin(2t)}$$

   b. (13 points) Evaluate the Laplace Transform

$$\mathcal{L}\{\,(t-3)\,\mathcal{U}(t-2) - (t-2)\mathcal{U}(t-3)\}$$

$$= \mathcal{L}\{(t-2-1)\mathcal{U}(t-2) - (t-2-1+1)\mathcal{U}(t-3)\}$$

$$= \mathcal{L}\{\underbrace{(t-2)\mathcal{U}(t-2)}_{f(t)=t \quad a=2} - \underbrace{\mathcal{U}(t-2)}_{a=2} - \underbrace{(t-3)\mathcal{U}(t-3)}_{f(t)=t \quad a=3} - \underbrace{\mathcal{U}(t-3)}_{a=3}\}$$

$$= \boxed{\frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-2s} - \frac{1}{s^2}e^{-3s} - \frac{1}{s}e^{-3s}}$$

MATH 420 – Differential Equations
Kristin Lui

3. (13 points) Find the general solution of the given system of equations below.

$$X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X$$

$$\det(\lambda I - A) = \begin{pmatrix} \lambda - 2 & 1 \\ -3 & \lambda+2 \end{pmatrix} = 0$$

$$\lambda^2 - 4 + 3 = 0$$

$$\lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda = \pm 1}$$

$$\lambda = 1: \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad k_1 = k_2 \quad \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

let $k_1 = 1$
$k_2 = 1$

$$\lambda = -1 \quad \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 3k_1 = k_2 \quad \vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

let $k_1 = 1$
$k_2 = 3$

$$\boxed{\vec{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}}$$

MATH 420 – Differential Equations
Kristin Lui

4. (20 points) Using the general solution from problem 3, find the particular solution for the system below.

$$X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$X_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \qquad \det(\Phi) = 3 - 1 = 2 \qquad \Phi^{-1}(t) = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

$$X_p = \Phi \int \Phi^{-1} \vec{F}(t)\, dt$$

$$= \Phi(t) \int \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} dt$$

$$= \frac{1}{2} \Phi(t) \int \begin{pmatrix} 3e^{-t}e^t + e^{-t} \cdot e^t \\ -e^t e^t - e^t e^t \end{pmatrix} dt$$

$$= \frac{1}{2} \Phi(t) \int \begin{pmatrix} 3+1 \\ -2e^{2t} \end{pmatrix} dt$$

$$= \frac{1}{2} \Phi \int \begin{pmatrix} 4 \\ -2e^{2t} \end{pmatrix} dt$$

$$= \frac{1}{2} \Phi \cdot \begin{pmatrix} 4t \\ -e^{2t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} 4t \\ -e^{2t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 4te^t - e^{-t}e^{2t} \\ 4te^t - 3e^{-t}e^{2t} \end{pmatrix}$$

$$= \boxed{\frac{1}{2} \begin{pmatrix} 4te^t - e^t \\ 4te^t - 3e^t \end{pmatrix}}$$

MATH 420 – Differential Equations
Kristin Lui

5. (15 points) For the DE below, find the singular points and classify the points as regular or irregular. Include justification for your classification.

$$x^2(1-x^2)y'' - \frac{2}{x}y' + 4y = 0$$

$$y'' - \frac{2}{x^3(1-x^2)}y' + \frac{4}{x^2(1-x^2)}y = 0$$

$$P(x) = -\frac{2}{x^3(1-x^2)} \qquad Q(x) = \frac{4}{x^2(1-x^2)}$$

Singular Points: $x^2(1-x^2)=0 \Rightarrow x_0 = 0$
$x_0 = \pm 1$

$x_0 = 0$ $\quad p(x) = x \cdot P(x) = \frac{-2}{x^2(1-x^2)}$ not analytic at $x_0 = 0$

$$\boxed{\Rightarrow x_0 = 0 \text{ is irregular singular}}$$

$x_0 = 1$ $\quad p(x) = (x-1)P(x) = \frac{-2}{x^3(x+1)}$ analytic at $x_0 = 1$

$q(x) = (x-1)^2 Q(x) = \frac{4(x-1)}{x^2(x+1)}$ analytic at $x_0 = 1$

$$\boxed{\Rightarrow x_0 = 1 \text{ is regular singular}}$$

$x_0 = -1$ $\quad p(x) = (x+1)P(x) = \frac{-2}{x^3(x-1)}$ analytic at $x_0 = -1$

$q(x) = (x+1)^2 Q(x) = \frac{4(x+1)}{x^2(x-1)}$ analytic at $x_0 = -1$

$$\boxed{\Rightarrow x_0 = -1 \text{ is regular singular}}$$

MATH 420 – Differential Equations
Kristin Lui

6. (12 points) Use the Laplace Transform to solve the initial value problem

$$y' + 4y = e^{2t} \qquad y(0) = 1$$

$$\mathcal{L}\{y' + 4y\} = \mathcal{L}\{e^{2t}\}$$

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s-2}$$

$$Y(s)[s+4] = 1 + \frac{1}{s-2}$$

$$Y(s) = \frac{1}{s+4} + \frac{1}{(s-2)(s+4)}$$

$a = 2$
$b = -4$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+4)}\right\}$$

$$y(t) = e^{-4t} + \frac{e^{2t} - e^{-4t}}{2+4}$$

$$y(t) = e^{-4t} - \frac{1}{6}e^{-4t} + \frac{1}{6}e^{2t}$$

$$\boxed{y(t) = \frac{5}{6}e^{-4t} + \frac{1}{6}e^{2t}}$$

Bonus (10 points) Evaluate the inverse Laplace Transform below.

$$\mathcal{L}^{-1}\left\{\frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-1)e^{-2s}}{s^2 - 2s + 1 + 1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-1)e^{-2s}}{(s-1)^2 + 1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{(s-1)e^{-2s}}{(s-1)^2 + 1}\right\}$$

$$F(s) = \frac{(s-1)}{(s-1)^2 + 1}$$

$$\rightarrow f(t) = e^{t}\cos(t)$$

by #19

$$= \cancel{2 \mathcal{L} \{ F(s) \}}$$

$$= \cancel{2 e^{t} \cos(t) u(t)}$$

sorry!

$$= \boxed{2 \cdot e^{(t-2)} \cos(t-2)\, u(t-2)}$$